ANOTHER LOOK AT FINDING CONTINUED FRACTIONS

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Abstract

The algorithm used to find the partial quotients of the continued fraction of an arbitrary real is closely related to the Euclidean algorithm. An alternative approach is presented which is analogous to converging sequences, which some may find more intuitive.

1. The Euclidean algorithm and continued fractions

One of the most well-known algorithms taught in discrete mathematics courses is the Euclidean algorithm to find the greatest common divisor of two integers p and q. The algorithms can be stated as follows: set $a_{-1} = p$, $a_0 = q$, then find b_i, a_{i+1} that satisfy $a_{i-1} = a_i b_i + a_{i+1}$ with $0 \le a_{i+1} < a_i$ for $i = 0, 1, \ldots$, until $a_{n+1} = 0$. The greatest common divisor is then a_n. The iterative step can be rewritten as $(a_{i-1}/a_i) = b_i + 1/(a_i/a_{i+1})$ and applied for $i = 0, 1, \ldots, n$ to give

$$\frac{p}{q} = b_0 + \cfrac{1}{b_1 + \cfrac{1}{b_2 + \cfrac{\vdots}{b_{n-1} + \cfrac{1}{b_n}}}} \equiv b_0 + \frac{1}{b_1 +}\,\frac{1}{b_2 +}\cdots\frac{1}{+\,b_n} \equiv [b_0; b_1, b_2, \ldots, b_n],$$

where b_0 is an integer and the b_i's for $i > 0$ are positive integers. This reformulation of the fraction p/q is known as a simple *continued fraction*, or SCF. More general continued fractions allow the numerators to be integers other than one, and can also allow the b_is to be negative or zero. The second and third forms are the most common abbreviations for SCFs in the literature, and the set $\{b_i\}$ are known as *partial quotients* of the SCF. If we identify a_{i-1}/a_i with x_i, then the Euclidean algorithm can be rewritten as the continued fraction algorithm: set $x_0 = x$ and then $b_i = \lfloor x_i \rfloor$, $x_{i+1} = 1/(x_i - b_i)$ for $i = 0, 1, \ldots, n$, where $\lfloor x \rfloor$ is the largest integer less than or equal to x. If x is irrational, then the iteration continues indefinitely, and we get an infinite continued fraction representation.

2. Some continued fraction properties

Almost every introductory number theory textbook (and some not so introductory!) includes a discussion of continued fractions to varying degrees of complexity. Every discussion of continued fractions includes the above algorithm for calculating partial quotients. As we shall see, there is an alternative approach, but first we should review some simple continued fraction properties. They will be presented without proof, but none are difficult, and can be found in, for example, Hardy and Wright [1].

The *convergents* of a continued fraction are the fractions $p_k/q_k = [b_0; b_1, \ldots, b_k]$ which approximate x (rational or irrational). They are in lowest form (numerator and denominator have no common factors), and can be found using the recursion $p_{-1} = 1$, $q_{-1} = 0$, $p_0 = b_0$, $q_0 = 1$, then $p_i = b_i p_{i-1} + p_{i-2}$, $q_i = b_i q_{i-1} + q_{i-2}$ for $i = 1, 2, \ldots$. An equivalent form of some elegance is $\begin{pmatrix} p_k & p_{k-1} \\ q_k & q_{k-1} \end{pmatrix} = \begin{pmatrix} b_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} b_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} b_k & 1 \\ 1 & 0 \end{pmatrix}$.

The convergents of a continued fraction converge to x and satisfy $|p_k/q_k - x| < 1/q_k^2$. The even convergents are increasing and approaching x from below, while the odd convergents are decreasing and approaching x from above i.e.

$$\frac{p_0}{q_0} < \frac{p_2}{q_2} < \frac{p_4}{q_4} < \cdots < \frac{p_{2i}}{q_{2i}} < \cdots < x < \cdots < \frac{p_{2i-1}}{q_{2i-1}} < \cdots < \frac{p_3}{q_3} < \frac{p_1}{q_1}.$$

Another way of putting it is that if $\alpha = [b_0; b_1, b_2, \ldots, b_{i-1}, a]$ and $\beta = [b_0; b_1, b_2, \ldots, b_{i-1}, b]$ where $a > b$, then $\alpha > \beta$ if i is even, and $\alpha < \beta$ if i is odd. A pair of convergents p_{i-1}/q_{i-1} and p_i/q_i bracket x, in the sense that one is a lower bound and the other is an upper bound. When x is rational, there are only a finite number of convergents, and one of the innermost inequalities is replaced by an equality. At this point I should point out that this bounding property of continued fraction convergents doesn't depend on calculating the b_i's for a given x. Any arbitrary sequence of b_i's will produce convergents that have the same sawtooth style of convergence, and can in fact be used as a limiting definition of a particular real number x.

Finally, the convergents of the continued fraction for x are a subset of the *best rational approximations* of x, defined as fractions p/q such that the error $|x - p/q|$ is smaller than that for any fraction with denominator smaller than q. More details on the relationship between best rational approximations and continued fractions can be found in Lucas and Lever [2]. The combination of this best approximation property with the rapidity of convergence explains the popularity of using continued fractions to represent reals.

3. Alternative derivation of partial quotients

Every text that develops the theory of continued fractions (at least those that the author has seen) derives the above algorithm for partial quotients based upon the Euclidean algorithm. To develop an alternative, consider the fraction $f(t) = (at + c)/(bt + d)$, where a/b and c/d are two fractions in lowest form. Obviously, $f(0) = c/d$ and $f(t) \to a/b$ as $t \to \infty$. Since $f'(t) = \left(\frac{a}{b} - \frac{c}{d}\right) \big/ (bd(bt + d)^2)$, if $a/b < c/d$, $f(t)$ is strictly decreasing, and if $a/b > c/d$ then $f(t)$ is strictly increasing.

If we now associate a/b with p_{i-1}/q_{i-1} and c/d with p_{i-2}/q_{i-2}, successive convergents of a continued fraction, then when t is an integer, $f(t)$ is the next convergent p_i/q_i of some number x between p_{i-1}/q_{i-1} and p_{i-2}/q_{i-2}. The actual choice of $t = b_i$ depends on x. Since $f(t)$ is a strictly monotonic function starting at p_{i-2}/q_{i-2} when $t = 0$ and moving towards p_{i-1}/q_{i-1}, and the number x is always bracketed by successive continued fraction convergents, we should choose t so that $f(t) = p_i/q_i$ is only just still on the same side of x as p_{i-2}/q_{i-2}. While smaller choices of t would also lead to a bracketed interval for x, it could mean at the next step we have to choose $t = 0$, which is illegal for a simple continued fraction.

Rosen [3] calls the numbers $(tp_{i-1}+p_{i-2})/(tq_{i-1}+q_{i-2})$ *pseudoconvergents*, and sets exercises to show that they are in lowest form (obvious since they are convergents of some number) and that those that are closer to x than previous convergents are best rational approximations.

More importantly, we can use these features of the fractions $f(t)$ and the bracketing nature of successive convergents to find the partial quotients of the continued fraction for x. Assume that the partial quotients $[b_0; b_1, \ldots, b_{n-1}]$ have been found. If n is even, then we want to find the integer b_n such that $p_n/q_n = (b_n p_{n-1} + p_{n-2})/(b_n q_{n-1} + q_{n-2})$ is $\leq x$ while $((b_n+1)p_{n-1} + p_{n-2})/((b_n+1)q_{n-1}+q_{n-2})$ is $> x$. The equality in the first case is to deal with possibly rational x. Equality also indicates the continued fraction is complete. If n is odd, then the inequalities are reversed. To start the process, b_0 is the largest integer less than x, and b_1 is the largest integer such that $b_0 + (1/b_1)$ is larger than x.

4. A simple example

Let us consider finding the continued fraction expansion for $\pi = 3.14159265358979\ldots$. The same process can be applied to any real.
The largest integer $< \pi$ is 3, so $b_0 = 3$, $p_0/q_0 = 3/1$.
$3 + 1/b_1$ is $> \pi$ when $b_1 < 1/(\pi - 3) = 7.0625\ldots$, so $b_1 = 7$, $p_1/q_1 = 3 + 1/7 = 22/7$.
The largest integer satisfying $(22b_2 + 3)/(7b_2 + 1) < \pi$ is $b_2 = 15$, so $p_2/q_2 = 333/106$.
The largest integer satisfying $(333b_3 + 22)/(106b_3 + 7) > \pi$ is $b_3 = 1$, so $p_3/q_3 = 355/113$.
The largest integer satisfying $(355b_4 + 333)/(113b_4 + 106) < \pi$ is $b_4 = 292$, so $p_4/q_4 = 103993/33102$.
The largest integer satisfying $(103993b_5 + 355)/(33102b_5 + 113) > \pi$ is $b_5 = 1$, so $p_5/q_5 = 104348/33215$.
The largest integer satisfying $(104348b_6 + 103993)/(33215b_6 + 33102) < \pi$ is $b_6 = 1$, so $p_6/q_6 = 208341/66317$, etc.

Note that as well as finding the partial quotients, the process also explicitly finds the convergents.

5. An algorithm

The inequalities that define b_n in section 3 above are equivalent to $b_n = \left\lfloor -\left(\frac{q_{n-2}}{q_{n-1}}\right)\left(\frac{e_{n-2}}{e_{n-1}}\right)\right\rfloor$, where $e_n = p_n/q_n - x$. The same formula applies for n both odd and even. We can this write an algorithm for this process as:

Set $b_0 = \lfloor x \rfloor$, $p_0 = b_0$, $q_0 = 1$, $c_0 = p_0$, $e_0 = c_0 - x$.
Set $b_1 = \lfloor x - b_0 \rfloor$, $p_1 = b_1 p_0 + 1$, $q_1 = b_1$, $c_1 = p_1/q_1$, $e_1 = c_1 - x$.
For $i = 2, n$
  Set $b_i = \lfloor -(q_{i-2}e_{i-2})/(q_{i-1}e_{i-1}) \rfloor$,
  Set $p_i = p_{i-1}b_i + p_{i-2}$, $q_i = q_{i-1}b_i + q_{i-2}$, $c_i = p_i/q_i$, $e_i = c_i - x$.
Endfor

The additional vectors c_i and e_i hold the partial fraction convergents and differences from x. As one would expect, this algorithm gives identical results to the standard algorithm.

On first impressions, this algorithm looks to be substantially less efficient than the standard continued fraction algorithm. Every iteration of the for loop requires one floor function, six multiplications/divisions, and three additions/subtractions (on typical machines, multiplications and divisions take roughly the same computational effort, as do additions and subtractions). This is far more than the effort expended by the standard algorithm described at the end of section 1, where in each iteration there is only one floor, one division, and one subtraction.

However, the alternative algorithm is not really that bad. In most situations where continued fractions are calculated, the convergents (c_i) and their errors ($|e_i|$) are also required. If this data is calculated as well as the partial quotients themselves, then the standard algorithm requires one floor, four multiplications/divisions and four additions/subtractions every iteration. While I am still not suggesting replacing the classical algorithm, this alternative version has the advantage of directly relating the continued fraction to fractions converging to x, particularly when described as in the example of section 3. Application of the standard algorithm, with its sequence of seemingly random x_i's, may conceal the true identity of the continued fraction convergents for some.

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References

[1] G.H. Hardy and E.M. Wright, *An introduction to the theory of numbers, 5th edition*, Oxford University Press, London, 1979.

[2] S.K. Lucas and K.V. Lever, On representing irrationals by continued fractions, *submitted*, (2002).

[3] K.H. Rosen *Elementary number theory and its applications, 3rd edition*, Addison-Wesley, Reading MA, 1992.