COMPUTATIONS FOR TIME-OPTIMAL BANG–BANG CONTROL USING A LAGRANGIAN FORMULATION [1]

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Abstract: In this paper an algorithm is proposed to solve the problem of time-optimal bang–bang control of nonlinear systems from a given initial state to a given terminal state. The problem is reduced to the problem of minimising a Lagrangian subject to an equality constraint defined by the terminal state. Then a solution is obtained by solving a system of nonlinear equations. Examples are given so as to illustrate the algorithm presented.

Keywords: Time-optimal control, Bang-bang control, Constraints, Control algorithms

1. INTRODUCTION

There have been a number of time-optimal bang–bang control algorithms reported in the literature, citations of which are given in (Kaya and Noakes, 1996) and (Scrivener and Thompson, 1994). The Switching-Time-Variation-Method (STVM) due to Mohler (1973; 1991), the Switch Time Optimization (STO) algorithm by Meier and Bryson (1990), an algorithm given by Teo *et al.* (1991) and Wong *et al.* (1985), the Control Parametrization Enhancing Technique (CPET) by Lee *et al.* (1997) by means of a general optimal control software called MISER, and Time-Optimal Switchings (TOS) due to Kaya and Noakes (Submitted) are immediate examples. In all of these algorithms the controls are assumed to be bang–bang, and the switching times can be calculated for a minimum terminal time.

The Switching Time Computations (STC) algorithm proposed by Kaya & Noakes (1996) finds a suitable concatenation of bang-arcs (or bang–bang trajectories) from an initial point to a target point for a given number of switchings. The solution found by STC is a feasible solution, which is not necessarily optimal. Lucas & Kaya (2001) presented a different numerical formulation and scheme for STC, which eliminated the need for the use of second-order variations. This formulation considered the problem of reaching from the initial to terminal state as the problem of solving a nonlinear system of equations, as opposed to a minimisation of the distance from the terminal point.

In this paper time-optimal bang–bang control of a nonlinear dynamical system from a given initial state to a given terminal state is considered. The problem is reduced to the problem of minimising a Lagrangian subject to an equality constraint defined by the terminal state. The Lagrangian minimisation problem itself reduces to solving a nonlinear system of equations, where the numerical scheme proposed in (Lucas and Kaya, 2001) is incorporated.

The TOS algorithm presented in (Kaya and Noakes, Submitted) needs a feasible bang–bang solution as the initial guess, which can typically be obtained using STC. Using the gradient calcu-

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lations as in (Kaya and Noakes, 1996) and the idea of sliding on the surface defined by the terminal state in the optimisation space, TOS achieves a time-optimal bang–bang solution. The algorithm proposed in this paper achieves the same task using second-order variations in addition to the gradient, however it does not require a feasible solution as the initial guess. In fact the initial guess can be very far from a feasible solution. In such situations the proposed algorithm is observed to handle the bad guess reasonably well.

After the description of the new algorithm, some example applications are given. Of these examples, the most notable one, namely the time-optimal control of F-8 aircraft, is shown to have a remarkably lower minimum than those reported in the literature.

Consider a general nonlinear system

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}, u), \tag{1}$$

where the state $\mathbf{x}(t) \in C([0,T];\mathbb{R}^n)$, the control $u(t)$ is a scalar piecewise constant function such that

$$u(t) = u_k, \quad \text{if } t \in [t_k, t_{k+1}).$$

Furthermore, $\mathbf{f}(\mathbf{x}, u) : R^n \times R^1 \to R^n$ is smooth in $\mathbf{x}$ for each value of u. The points t_k ($k = 1, 2, \ldots$) where $u(t)$ is discontinuous are called the *switching times*. Let N be the number of switchings taking place in the interval (t_0, t_f), so

$$t_0 < t_1 \leq \cdots \leq t_N < t_f.$$

The control u is called admissible (Pontryagin *et al.*, 1962) for the pre-specified initial and terminal points $\mathbf{x}_0$ and $\mathbf{x}_T$ if (1) results in a solution satisfying

$$\mathbf{x}(t_0) = \mathbf{x}_0 \quad \text{and} \quad \mathbf{x}(t_f) = \mathbf{x}_T.$$

One can construct an admissible control u with u_k, $k = 1, \ldots, N+1$, by appropriately choosing the switching times $t_1, \ldots, t_N$ and the final time t_f. We will the use the conventional abbreviation STC (e.g. (Kaya and Noakes, 1996)) to refer to such switching time computation problems.

A segment of the trajectory $\mathbf{x}(t)$ corresponding to the time interval from t_{k-1} to t_k represents a smooth arc. The dynamical system (1) can also be written as a sequence of initial value problems

$$\begin{aligned} \frac{d\mathbf{x}}{dt} &= \mathbf{f}_k(\mathbf{x}),\ t \in (t_{k-1}, t_{k-1} + \xi_k), \\ \mathbf{x}(t_{k-1}) &= \begin{cases} \mathbf{x}_0, & \text{if } k = 1, \\ \mathbf{x}(t_{k-1} - 0), & \text{if } k > 1, \end{cases} \end{aligned} \tag{2}$$

where ξ_k is the time-duration of the k-th arc, or simply the k-th *arc-time*, given by

$$\xi_k = \begin{cases} t_k - t_{k-1}, & \text{if } k = 1, 2, \ldots N, \\ t_f - t_N, & \text{if } k = N+1. \end{cases}$$

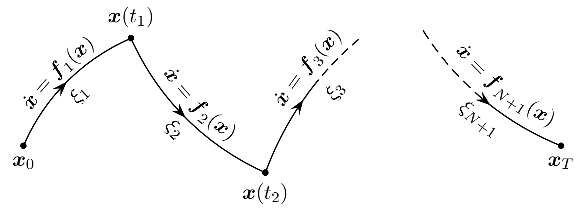

Fig. 1. An admissible trajectory from $\mathbf{x}_0$ to $\mathbf{x}_T$.

A sketch of a trajectory for an admissible control is shown in Figure 1. We will call such a trajectory an *admissible trajectory*.

The STC problem is usually formulated in terms of arc-times as in (Kaya and Noakes, 1996). The segment of a trajectory $\mathbf{x}(t)$ corresponding to the interval $[t_{k-1}, t_k]$ can be parametrised as $\mathbf{x}(t_{k-1} + \tau)$, where $\tau \in [0, \xi_k]$ ($k = 1, \ldots, N+1$). We will use the notation

$$\mathbf{x}(\tau; \xi_{k-1}, \ldots, \xi_1) \equiv \mathbf{x}(t_{k-1} + \tau)\,, \tag{3}$$

which explicitly shows that the behaviour of $\mathbf{x}$ in the k-th arc also depends on the previous arc-times. Note that for the first arc this notation simply becomes $\mathbf{x}(\tau)$ as there are no previous arcs.

Now the STC problem can be formulated as follows:

$$P_{\text{STC}} : \begin{cases} \text{given } \mathbf{x}_0 \text{ and } \mathbf{x}_T \text{ determine} \\ \text{non-negative } \{\xi_i\} \text{ such that} \\ \quad \mathbf{x}(\xi_{N+1}; \xi_N, \ldots, \xi_1) = \mathbf{x}_T\,. \end{cases} \tag{4}$$

We assume that in general $N + 1 \geq n$, otherwise the system (4) is overdetermined. If $N + 1 = n$, then the number of equations in (4) is the same as the number of variables and we could expect a locally unique solution. One of the techniques for numerical solution of nonlinear systems of equations can be employed for the solution of (4). A solution to Problem (P_{STC}) is reported in (Lucas and Kaya, 2001).

In this paper in taking $N + 1 \geq n$, the extra arcs will be used particularly for finding time-optimal solutions of (4), i.e. the solutions for which the total time

$$t_f = \xi_1 + \xi_2 + \cdots + \xi_{N+1}$$

is minimised. The technique we present is applicable to more general cost functionals given in the form

$$W = \sum_{i=1}^{N+1} \int_0^{\xi_i} g_i(\mathbf{x}(\tau; \xi_{i-1}, \ldots, \xi_1))d\tau. \tag{5}$$

In particular, if $g_i = 1$, $i = 1, \ldots, N+1$, we have the time-optimal problem. If $g_i(\mathbf{x}) = |f_i(\mathbf{x})|$, the objective function is the total length of the trajectory. We will focus our attention on the time-optimal bang–bang control problem, even though we will pose the technique for the general form in (5).

In Section 2 we give the description of an optimisation procedure that first performs a reduction to a minimisation problem with equality constraints, which is then treated further using the Lagrange multipliers technique.

2. REDUCTION TO A MINIMISATION WITH EQUALITY CONSTRAINTS

It will be assumed throughout that the following are specified:

- N, the number of switchings;
- $\{u_k\}$ $(k = 1, \ldots, N+1)$, values of $u(t)$ in respective arcs;
- $\mathbf{x}_0$ and $\mathbf{x}_T$, initial and target points.

Each possible control $u(t)$ is defined by a combination of positive $\{\xi_k\}$ $(k = 1, \ldots, N+1)$, hence the cost functional (5) acting on such controls is a function of $(\xi_1, \ldots, \xi_{N+1})$. Our aim is to develop a procedure for minimising $W(\xi_1, \ldots, \xi_{N+1})$ subject to the constraints:

$$\begin{aligned} &\mathbf{x}(\xi_{N+1}; \xi_N, \ldots, \xi_1) = \mathbf{x}_T, \\ &\xi_i \geq 0,\ i = 1, \ldots, N+1. \end{aligned}$$

First we introduce new variables $\{\alpha_i\}$ such that

$$\xi_i = \alpha_i^2, \quad (i = 1, \ldots, N+1)\,.$$

The minimisation problem formulated using α_i will not involve inequality constraints as the resulting ξ_i will be always nonnegative.

We will use the notation:

$$\begin{aligned} \boldsymbol{\alpha} \equiv &(\alpha_1, \ldots, \alpha_{N+1})^T; \\ \mathbf{x}(\boldsymbol{\alpha}) \equiv &\, \mathbf{x}(\xi_{N+1}; \xi_N, \ldots, \xi_1)\Big|_{\xi_i = \alpha_i^2; i=1,\ldots,N+1}; \\ w(\boldsymbol{\alpha}) \equiv &\, W(\xi_1, \ldots, \xi_{N+1})\Big|_{\xi_i = \alpha_i^2; i=1,\ldots,N+1}. \end{aligned}$$

The general optimal switching time computation problem can now be formulated as follows:

$$P_{\mathrm{OSC}} : \begin{cases} \text{minimise} & w(\boldsymbol{\alpha}) \\ \text{subject to} & \mathbf{x}(\boldsymbol{\alpha}) = \mathbf{x}_T. \end{cases} \tag{6}$$

The form of $w(\boldsymbol{\alpha})$ in Problem (P_{OSC}) can be easily derived from (5). In particular, for the time-optimal problem we have $w(\boldsymbol{\alpha}) = \boldsymbol{\alpha}^T \boldsymbol{\alpha}$.

Problem (6) is a standard optimisation problem with equality constraints, its Lagrangian has the form

$$L(\boldsymbol{\alpha}; \boldsymbol{\lambda}) = w(\boldsymbol{\alpha}) + (\mathbf{x}(\boldsymbol{\alpha}) - \mathbf{x}_T)^T \boldsymbol{\lambda}\,,$$

where $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)^T$.

The Lagrange conditions are

$$\begin{cases} \nabla_{\boldsymbol{\alpha}} w + J^T \boldsymbol{\lambda} = \mathbf{0}, \\ \mathbf{x}(\boldsymbol{\alpha}) - \mathbf{x}_T = \mathbf{0}, \end{cases} \tag{7}$$

where

$$J = \begin{pmatrix} x_{1\alpha_1} & \cdots & x_{1\alpha_{N+1}} \\ \vdots & \vdots & \vdots \\ x_{n\alpha_1} & \cdots & x_{n\alpha_{N+1}} \end{pmatrix}, \quad x_{i\alpha_j} \equiv \frac{\partial x_i}{\partial \alpha_j}\,. \tag{8}$$

The system in (7) consists of $N+1+n$ equations in $N+1+n$ unknown components of $\boldsymbol{\alpha}$ and $\boldsymbol{\lambda}$.

A numerical solution of the Lagrange equations (7) can be obtained using one or another modification of Newton's method.

If equations (7) are satisfied at a point $\boldsymbol{\alpha}_0$, then $\boldsymbol{\alpha}_0$ is a possible minimiser. Further investigation of the behaviour of the Lagrange function at this point involves examination of the quadratic form

$$\sum_{i=1}^{N+1} \sum_{j=1}^{N+1} L_{\alpha_i \alpha_j}(\boldsymbol{\alpha}_0) d\alpha_i d\alpha_j, \tag{9}$$

where $L_{\alpha_i \alpha_j} = \dfrac{\partial^2}{\partial \alpha_i \partial \alpha_j} L$.

In (9) only $N+1-n$ differentials are independent as $d\boldsymbol{\alpha}$ must satisfy

$$J(\boldsymbol{\alpha}_0) d\boldsymbol{\alpha} = \mathbf{0}, \tag{10}$$

where J is given by (8). System (10) is a direct consequence of the equality constraints $\mathbf{x}(\boldsymbol{\alpha}) = \mathbf{x}_T$. Using (10) and expressing n dependent differentials in terms of the independent differentials and substituting the result into (9) we obtain a quadratic form in restricted variables. If it turns out that this form is positive definite then $\boldsymbol{\alpha}_0$ is a local minimiser for Problem (P_{OSC}) in (6).

Let us describe a computationally straightforward post-processing procedure that allows us to determine the sign of the quadratic form (9) under the constraints (10). Introduce the notation:

$$\begin{aligned} H = &[L_{\alpha_i \alpha_j}(\boldsymbol{\alpha}_0)]; \\ \boldsymbol{\beta} = &(\beta_1, \beta_2, \ldots, \beta_{N+1})^T \\ = &(d\alpha_1, d\alpha_2, \ldots, d\alpha_{N+1})^T. \end{aligned}$$

Rewrite equations (9) and (10) in this notation and consider

$$\boldsymbol{\beta}^T H \boldsymbol{\beta} \quad \text{subject to } J(\boldsymbol{\alpha}_0)\boldsymbol{\beta} = \mathbf{0}. \tag{11}$$

We assume that conditions $\mathbf{x}(\boldsymbol{\alpha}) = \mathbf{x}_T$ are independent and therefore the rank of $J(\boldsymbol{\alpha}_0)$ is n. Let B be an $n \times n$ matrix formed by n linearly independent columns of $J(\boldsymbol{\alpha}_0)$ and $\hat{\boldsymbol{\beta}}$ be a vector made of the corresponding β_i. Similarly, let G be an $n \times \nu$ matrix formed by the remaining $\nu = N + 1 - n$ columns of $J(\boldsymbol{\alpha}_0)$ and $\tilde{\boldsymbol{\beta}}$ be a vector made of the components of $\boldsymbol{\beta}$ with the corresponding subscripts. Use the equivalence

$$J(\boldsymbol{\alpha}_0)\boldsymbol{\beta} = \mathbf{0} \quad \Leftrightarrow \quad B\hat{\boldsymbol{\beta}} = -G\tilde{\boldsymbol{\beta}}$$

to express $\hat{\boldsymbol{\beta}}$ in terms of $\tilde{\boldsymbol{\beta}}$: $\hat{\boldsymbol{\beta}} = A\tilde{\boldsymbol{\beta}}$, where $A = -B^{-1}G$. Taking the permutation matrix P such that

$$\boldsymbol{\beta} = P \begin{bmatrix} \hat{\boldsymbol{\beta}} \\ \tilde{\boldsymbol{\beta}} \end{bmatrix},$$

and substituting into the form in (11) we obtain

$$\begin{aligned} \boldsymbol{\beta}^T H \boldsymbol{\beta} &= \left[\hat{\boldsymbol{\beta}}^T \tilde{\boldsymbol{\beta}}^T\right] P^T H P \begin{bmatrix} \hat{\boldsymbol{\beta}} \\ \tilde{\boldsymbol{\beta}} \end{bmatrix} \\ &= \tilde{\boldsymbol{\beta}}^T \left[A^T | I_{\nu\times\nu}\right] P^T H P \left[\frac{A}{I_{\nu\times\nu}}\right] \tilde{\boldsymbol{\beta}} \\ &= \tilde{\boldsymbol{\beta}}^T Q \tilde{\boldsymbol{\beta}}, \end{aligned}$$

our quadratic form in restricted variables. Positive-definiteness of this form is a sufficient condition for $\boldsymbol{\alpha}_0$ to be a local minimiser of $w(\boldsymbol{\alpha}_0)$.

3. NUMERICAL DETAILS

We first combine the left-hand sides of the system in (7) into a single vector $\boldsymbol{\Phi}(\boldsymbol{\alpha}, \boldsymbol{\lambda})$ and rewrite the Lagrange conditions in the form

$$\boldsymbol{\Phi}(\boldsymbol{\alpha}, \boldsymbol{\lambda}) = \mathbf{0}. \tag{12}$$

When applying the standard Newton's method to solving (12), the next iterate after $(\boldsymbol{\alpha}, \boldsymbol{\lambda})$ is $(\boldsymbol{\alpha} + \delta\boldsymbol{\alpha}, \boldsymbol{\lambda} + \delta\boldsymbol{\lambda})$ where $(\delta\boldsymbol{\alpha}, \delta\boldsymbol{\lambda})$ are found from the linear system

$$J_{\boldsymbol{\Phi}}(\boldsymbol{\alpha}, \boldsymbol{\lambda}) \begin{pmatrix} \delta\boldsymbol{\alpha} \\ \delta\boldsymbol{\lambda} \end{pmatrix} = -\boldsymbol{\Phi}(\boldsymbol{\alpha}, \boldsymbol{\lambda}),$$

in which $J_{\boldsymbol{\Phi}}$ is the Jacobian matrix of the form

$$J_{\boldsymbol{\Phi}} \equiv \left(\begin{array}{c|c} [L_{\alpha_i\alpha_j}] & J^T \\ \hline J & 0_{n\times n} \end{array} \right). \tag{13}$$

Note that more robust convergence from arbitrary initial guesses is obtained using a modified Newton's method, where one may only use a fraction of the update vector. In (13) $[L_{\alpha_i\alpha_j}]$ is the hessian of the Lagrange function with respect to $\boldsymbol{\alpha}$ and J is given by (8). Note that

$$J = 2 J_{\boldsymbol{\xi}} \,\mathrm{diag}(\alpha_1, \ldots,\ \alpha_{N+1}), \tag{14}$$

where $J_{\boldsymbol{\xi}}$ is the Jacobian of $\mathbf{x}(\xi_{N+1}; \xi_N, \ldots, \xi_1)$. Matrix $J_{\boldsymbol{\xi}}$ can be evaluated through numerical solution of the systems of ordinary differential equations derived from (2) by differentiating in respective variables (as in e.g. (Kaya and Noakes, 1996)).

The hessian of the Lagrange function used in (9) and in (13) can be written as

$$[L_{\alpha_i\alpha_j}] = \left(\frac{\partial \boldsymbol{\phi}}{\partial \alpha_1}, \ldots, \frac{\partial \boldsymbol{\phi}}{\partial \alpha_{N+1}} \right), \tag{15}$$

where $\boldsymbol{\phi}(\boldsymbol{\alpha}, \boldsymbol{\lambda}) \equiv \nabla_{\boldsymbol{\alpha}} w + J^T \boldsymbol{\lambda}$. It turns out that for the iterative procedure that uses (13) it is quite acceptable to evaluate (15) using central difference quotients. Care must be taken however if the hessian is needed to verify sufficient conditions for a minimum. In particular, the described method of evaluation of the hessian does not guarantee that the result is a symmetric matrix. Furthermore, the technique of computation of the Jacobian using (14) and the system derived from (2) by differentiating in ξ_i may turn out to be unsatisfactory at points where the hessian has large values. In such cases, once a moderate proximity to a possible solution has been reached, we could switch to a slower method of Jacobian evaluation using finite differences. This kind of difficulty is not present in the methods which do not use the Jacobian in the target equations (Kaya and Noakes, Submitted; Lucas and Kaya, 2001).

In (Kaya and Noakes, 1996), second-order variations of the terminal state with respect to arc-times are incorporated in the basic optimization routines, Newton's method and steepest descent. However in (Lucas and Kaya, 2001) through a different formulation and numerical scheme this need for second-order variations is eliminated. One should note that both of these papers (Kaya and Noakes, 1996; Lucas and Kaya, 2001) present methods for finding feasible, but not necessarily optimal, solutions. In (Kaya and Noakes, Submitted) no second-order variations are being used, and furthermore the method presented solves the time-optimal control problem.

4. NUMERICAL EXAMPLES

4.1 *The F-8 aircraft examples*

We will discuss now the results of application of our technique to the study of the dynamical model

$$\begin{aligned} \dot{x}_1 =& -0.877x_1 + x_3 - 0.088x_1x_3 + 0.47x_1^2 \\ & - 0.019x_2^2 - x_1^2x_3 + 3.846x_1^3 - 0.215u \\ & + 0.28x_1^2u + 0.47x_1u^2 + 0.63u^3, \\ \dot{x}_2 =& x_3, \\ \dot{x}_3 =& -4.208x_1 - 0.396x_3 - 0.47x_1^2 \\ & - 3.564x_1^3 - 20.967u + 6.265x_1^2u \\ & + 46x_1u^2 + 61.4u^3, \end{aligned} \tag{16}$$

which governs behaviour of the F-8 aircraft (Garrard and Jordan, 1977). System (16) has become a traditional testing ground for various optimal control strategies, so, for convenience of comparison, we will use the standard settings (Kaya and Noakes, Submitted; Lee *et al.*, 1997)

$$\mathbf{x}_0 = \frac{\pi}{180}(26.7,\, 0,\, 0)^T,\ \mathbf{x}_T = (0,\, 0,\, 0)^T,$$
$$u = \pm 0.05236\,.$$

Since the Lagrange equations are the conditions for a local minimiser, computation results depend on the initial guess, which requires specification of both the arc-times and the Lagrange coefficients. We organise various initial guesses and corresponding results in separate examples. The quadratic form tests applied to these examples indicated that the resulting points were minima.

Example F-8-1. First we consider the following result for a 4-arc configuration with $u(0) = 0.05236$:

	Seed	Result
ξ_1	1	1.132765
ξ_2	0.8	0.347492
ξ_3	1	1.608881
ξ_4	1	0.692379

	Seed	Result
λ_1	0	2.322838
λ_2	0	-1.396123
λ_3	0	-0.942077

Total time: 3.781517.

The obtained total time of 3.781517 is significantly smaller than the 6-arc solution of 5.742177 given in (Kaya and Noakes, Submitted) and the 4-arc solution of 6.035 in *et al.* (Lee *et al.*, 1997). The program required 34 iterations to obtain the above arc-times and Lagrange coefficients, which satisfy equations (7) with an accuracy of order 10^{-6} (in nearness to the target $(0,0,0)^T$). As one would expect, accuracies of order less than 10^{-6} can be achieved with a fewer extra iterations.

Example F-8-2. The same minimiser can be obtained from the below-listed five arcs ($u(0) = 0.05236$) being used as an initial guess:

	Seed	Result
ξ_1	1	1.132765
ξ_2	0.3	0.347492
ξ_3	1.5	1.608881
ξ_4	1	0.692379
ξ_5	1	0

	Seed	Result
λ_1	0	2.322838
λ_2	0	-1.396123
λ_3	0	-0.942077

Total time: 3.781517

The same tolerance of 10^{-6} has been used for this example. The final arc configuration is reached for 33 iterations. The first 4 arcs in this result are the same as those obtained in the example F-8-1, the last arc has been eliminated.

Thus, in this case, the total arc-time as a function of ξ_i $(i = 1,\ldots,5)$ has the same constrained minimum as its restriction to the hyperplane of the first four arc-time variables.

Example F-8-3. Here is the result for 6 arcs ($u(0) = 0.05236$):

	Seed	Result
ξ_1	1	1.1327648
ξ_2	1	0.3474915
ξ_3	1	1.6088814
ξ_4	1	0.2223491
ξ_5	1	0
ξ_6	1	0.4700298

	Seed	Result
λ_1	0	2.322838
λ_2	0	-1.396123
λ_3	0	-0.942077

Total time: 3.781517.

The combination of the last three arcs in this result are equivalent to one arc, as its middle arc has been eliminated. The sum of the remaining two arc-times has the same value as the value of the fourth arc-time in examples F-8-1 and F-8-2. Though the same tolerance of 10^{-6} has been used for this example we provide more digits in final values of arc-times to offset the effect of rounding errors. The final arc configuration is reached in 29 iterations.

A different choice of initial arc-times and Lagrange coefficients can yield a different minimum time as illustrated in the following example.

Example F-8-4. Consider the following result for 6 arcs ($u(0) = 0.05236$):

	Seed	Result
ξ_1	0.5	0.102917
ξ_2	1	1.927923
ξ_3	0.5	0.166868
ξ_4	1	2.743384
ξ_5	0.5	0.329923
ξ_6	0.5	0.471162

	Seed	Result
λ_1	0	10.951790
λ_2	0	-7.673815
λ_3	0	-1.030559

Total time: 5.742177.

This result required 9 iterations. The arc times (and so the total time) are the same as those reported in Kaya & Noakes (Submitted) for a similar configuration.

Remark 1. It is interesting to note that the time-optimal bang–bang solutions for the F-8 aircraft presented in (Kaya and Noakes, Submitted), (Lee *et al.*, 1997) and this work give three different local optima. These local solutions are summarised below for comparison.

- Reference (Lee *et al.*, 1997):

	Result
ξ_1	2.188
ξ_2	0.164
ξ_3	2.881
ξ_4	0.330
ξ_5	0.472

$u(0) = -0.05236$

$t_f = 6.035$

- Reference (Kaya and Noakes, Submitted):

	Result
ξ_1	0.10292
ξ_2	1.92793
ξ_3	0.16687
ξ_4	2.74338
ξ_5	0.32992
ξ_6	0.47116

$u(0) = 0.05236$

$t_f = 5.74217$

- This paper (see example F-8-1):

$$u(0) = 0.05236, \quad t_f = 3.781517$$

It is conceivable to think that these are only a few of the many local minima for the terminal time. It would be worthwhile to try and find other possible local minima.

4.2 *Examples for the van der Pol equation and a third order nonlinear system*

The van der Pol equation can be reduced to the system

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -x_1 - (x_1^2 - 1)x_2 + u(t),\ u = \pm 1. \end{aligned} \tag{17}$$

Example vdP-1. Consider the initial and terminal points (Kaya and Noakes, 1996)

$$\mathbf{x}_0 = (-0.4, -0.6)^T, \quad \mathbf{x}_T = (0.6, 0.4)^T.$$

Results for seven arcs (u(0)=1):

	Seed	Result
ξ_1	0.3	0.9751925
ξ_2	0.3	0
ξ_3	0.3	0.0006058
ξ_4	0.3	0
ξ_5	0.3	0.0008364
ξ_6	0.3	1.1637249
ξ_7	0.3	0

	Seed	Result
λ_1	0	−0.437765
λ_2	0	0.613760

Total time: 2.140360

Required number of iterations is 35. Furthermore

$$t_f = \xi_1 + \xi_3 + \xi_5 = 0.976635,$$

which is in agreement with (Kaya and Noakes, 1996).

Example vdP-2. Consider the initial and terminal points (Kaya and Noakes, Submitted)

$$\mathbf{x}_0 = (1, 1)^T, \quad \mathbf{x}_T = (0, 0)^T.$$

Results for six arcs ($u(0) = -1$):

	Seed	Result
ξ_1	0.5	0.6908253
ξ_2	0.5	0
ξ_3	0.5	0.0321784
ξ_4	0.5	1.1752145
ξ_5	0.5	0
ξ_6	0.5	1.1969841

	Seed	Result
λ_1	0	0.196691
λ_2	0	−1

Total time: 3.095202

Required number of iterations is 22. Furthermore

$$\begin{aligned} \xi_1 + \xi_3 &= 0.723004, \\ \xi_4 + \xi_6 &= 2.372199, \end{aligned}$$

which is in agreement with (Kaya and Noakes, Submitted).

However, the current implementation cannot handle some difficult initial guesses from (Kaya and Noakes, Submitted).

Now consider the following third-order nonlinear system:

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= x_3, \\ \dot{x}_3 &= \sin(x_1) + u(t), \quad u(t) = \pm 1. \end{aligned} \tag{18}$$

Example ns-3-1. Consider the initial and terminal points (Kaya and Noakes, Submitted)

$$\mathbf{x}_0 = (1, 1, 1)^T, \quad \mathbf{x}_T = (0, 0, 0)^T.$$

Results for five arcs ($u(0) = -1$):

	Seed	Result
ξ_1	3.4276	3.538356
ξ_2	3.5911	3.598507
ξ_3	2.0896	1.1152856
ξ_4	0.7055	0
ξ_5	1.0927	0.8370915

	Seed	Result
λ_1	0	−0.908375
λ_2	0	−0.141088
λ_3	0	1

Total time: 9.089241

The process converges in 16 iterations. Since

$$\xi_3 + \xi_5 = 1.952377,$$

this result is in agreement with the arc times reported in (Kaya and Noakes, Submitted):

$$\boldsymbol{\xi} = (3.53835, 3.59851, 0, 0, 1.95238)^T.$$

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