Integrals Approximating π with Non-negative Integrands

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Motivational Example

Consider $I = \int_0^1 \frac{x^4(1-x)^4}{1+x^2}\,dx$.

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Originally described by Dalzell (1944).

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or

$$\frac{1979}{630} = \frac{22}{7} - \frac{1}{630} < \pi < \frac{22}{7} - \frac{1}{1260} = \frac{3959}{1260}.$$

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A Series Expansion for π

The partial fraction decomposition: $\dfrac{4}{1+x^2} = \dfrac{x^6 - 4x^5 + 5x^4 - 4x^2 + 4}{1 + x^4(1-x)^4/4}$.

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$$\pi = \sum_{k=0}^{\infty} \left(-\frac{1}{4}\right)^k \int_0^1 (x^6 - 4x^5 + 5x^4 - 4x^2 + 4)x^{4k}(1-x)^{4k}\, dx.$$

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For nonnegative integers m and n, $\displaystyle\int_0^1 x^m(1-x)^n\,dx = \frac{m!n!}{(m+n+1)!}$.

So...

Series Expansion (cont'd)

$$\pi = \sum_{k=0}^{\infty}\left(-\frac{1}{4}\right)^k \left[\frac{(4k)!(4k+6)!}{(8k+7)!} - \frac{4(4k)!(4k+5)!}{(8k+6)!} + \frac{5(4k)!(4k+4)!}{(8k+5)!} - \frac{4(4k)!(4k+2)!}{(8k+3)!} + \frac{4(4k)!^2}{(8k+1)!}\right]$$

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$$= \sum_{k=0}^{\infty}(-1)^k 4^{2-k}\frac{(4k)!(4k+3)!}{(8k+7)!}(820k^3 + 1533k^2 + 902k + 165)$$

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Series Expansion (cont'd)

$$
\begin{aligned}
\pi &= \sum_{k=0}^{\infty} \left(-\frac{1}{4}\right)^{k} \left[\frac{(4k)!(4k+6)!}{(8k+7)!} - \frac{4(4k)!(4k+5)!}{(8k+6)!} + \frac{5(4k)!(4k+4)!}{(8k+5)!} \right. \\
&\qquad \left. - \frac{4(4k)!(4k+2)!}{(8k+3)!} + \frac{4(4k)!^2}{(8k+1)!}\right] \\
&= \sum_{k=0}^{\infty} (-1)^k 4^{2-k} \frac{(4k)!(4k+3)!}{(8k+7)!}(820k^3 + 1533k^2 + 902k + 165) \\
&= \frac{22}{7} - \frac{19}{15015} + \frac{543}{594914320} - \frac{77}{104187267600} + \cdots .
\end{aligned}
$$

Series Expansion (cont'd)

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Roughly three digits of accuracy are added per term.

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More General Integrals

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$$\frac{x^{4n}(1-x)^{4n}}{1+x^2} = (x^6-4x^5+5x^4-4x^2+4)\sum_{k=0}^{n-1}(-4)^{n-1-k}x^{4k}(1-x)^{4k} + \frac{(-4)^n}{1+x^2}$$

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so integrating and simplifying,

$$\frac{(-1)^n}{4^{n-1}}\int_0^1 \frac{x^{4n}(1-x)^{4n}}{1+x^2}\,dx = \pi - \sum_{k=0}^{n-1}(-1)^k \frac{2^{4-2k}(4k)!(4k+3)!}{(8k+7)!}\times$$

$$(820k^3+1533k^2+902k+165)$$

Pancake Functions

(Backhouse 1995) $I_{m,n} = \int_0^1 \frac{x^m(1-x)^n}{1+x^2}\,dx = a + b\pi + c\ln(2)$, where rational a, b, c depend on positive integers m and n, and a and b have opposite sign.

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If $2m - n \equiv 0(\text{mod } 4)$, $c = 0$ and we get a π approximation.

Using $1 < 1 + x^2 < 2$ for $x \in (0,1)$ we get the error bounds

$$\frac{m!n!}{2(m+n+1)!} < a + b = \int_0^1 \frac{x^m(1-x)^n}{1+x^2}\,dx < \frac{m!n!}{(m+n+1)!}.$$

New Series Expansions

Applying the series expansion process to I_{4n} leads to

$$\pi = \sum_{m=0}^{\infty}\sum_{k=0}^{n-1}(-4)^{-nm-k}\int_0^1 (x^6 - 4x^5 + 5x^4 - 4x^2 + 4)\times x^{4(k+nm)}(1-x)^{4(k+nm)}\,dx$$

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$n=1$ leads to previous series expansion.

New Series (cont'd)

With $n = 2$,

$$\pi = \sum_{m=0}^{\infty} 4^{2-2m} \left[\frac{(8m)!(8m+3)!}{(16m+7)!}(6560m^3 + 6132m^2 + 1804m + 165) - \frac{(8m+4)!(8m+7)!}{(16m+15)!}(1640m^3 + 3993m^2 + 3214m + 855)\right]$$

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We can take n as large as we like, and produce series that add roughly $3n$ digits per term!

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Other Integral Approximations

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It is one of the continued fraction *convergents* of π, which begin 3, $22/7$, $333/106$, $355/113$, $103993/33102$, and $104348/33215$.

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Unfortunately, none of the integrals considered so far lead to these fractions.

Consider the slightly more complicated

$$\int_0^1 \frac{x^m(1-x)^n(a+bx+cx^2)}{1+x^2}\,dx = \alpha + \beta\pi + \gamma\ln 2,$$

where α, β and γ depends on m, n, a, b and c.

Approximation Algorithm

To set $\int_0^1 \frac{x^m(1-x)^n(a+bx+cx^2)}{1+x^2}\,dx = z - \pi$ or $\pi - z$ for a given z:

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- Choose m and n large enough that $a + bx + cx^2 \geq 0$ for $x \in [0, 1]$.
- Choose m and n as small as possible to make a, b and c as simple as possible.

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$$\int_0^1 \frac{x^{12}(1-x)^{12}(1349-1060x^2)}{38544(1+x^2)}\,dx = \frac{104348}{33215} - \pi.$$

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Conclusion

- We have developed a new family of series for π where each term can add $3n$ digits of accuracy for any n.
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Further Directions

- Are other integrands worth considering? I've already tried
$$\int_0^1 \frac{x^m(1-x)^n}{\sqrt{1-x^2}}\,dx.$$
- Can we approximate other constants this way?