The Power Series Method for Odes

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Outline

- The Taylor method and issues.

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- Formal series substitution.

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- Future Directions.

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The Taylor method is the first taught in a numerical ode class. But even simple right hand sides require lots of work, so Runge-Kutta, multistep etc.

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y'' &= \frac{d}{dt}\left(-x + y - x^2 y\right) = -x' + y' - 2xx'y - x^2 y' \\
&= -y - x + y - x^2 y - 2xy^2 - x^2(-x + y - x^2 y) \\
&= -x - 2xy^2 - 2x^2 y + x^3 + x^4 y,
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Ouch! And don't even think about $x' = \cos\left(1 + e^{3 \sin x}\right)$.

Another Ode Solution Technique

A first course in differential equations introduces a power series substitution method for second order linear differential equations of the from $p(x)y'' + q(x)y' + r(x)y = f(x)$, as long as p, q, r, f are sufficiently simple.

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Almost never seen again, especially when an ode is nonlinear. But, there is no reason why it can't be applied to other odes...

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Back to Van der Pol

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$y' = -x + y - x^2 y$ becomes

$$\sum_{i=1}^{\infty} i y_i t^{i-1} = -\sum_{i=0}^{\infty} x_i t^i + \sum_{i=0}^{\infty} y_i t^i - \left(\sum_{i=0}^{\infty}\left(\sum_{j=0}^{i} x_j x_{i-j}\right) t^i\right)\left(\sum_{i=0}^{\infty} y_i t^i\right),$$

and equate coefficients.

Even Better Van der Pol

Let $u = x^2$, then $x' = y$, $y' = -x + y - uy$, $x(0) = x_0$, $y(0) = y_0$, $u(0) = x_0^2$.

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Great for polynomial right hand side, but isn't that a major restriction?

A Bold Claim

- **Any** system of odes with analytic solutions, or analytic functions that can be represented as solutions of odes, can be reformulated in polynomial form – mostly as $y' = f(t, y)$, sometimes as $zy' = f(t, y)$ where z is a vector of ones or t's (regular singular points).

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- A systematic approach can be applied to make the conversion to polynomial form, and identify intermediate variables.

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- Extensive theoretical background (Ed Parker, Dave Carothers).

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If $y' = \sin z$, let $u_1 = \sin z$ and $u_2 = \cos z$, then $y' = u_1$, $u_1' = u_2 z'$ and $u_2' = -u_1 z'$.

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If $y = 1/f(t)$ where f has a known Taylor series, let $z = f'/f$, then
$y' = -f'/f^2 = -yz$ and $z' = (ff'' - f'^2)/f^2 = (f''/f) - (f'/f)^2$
$= yf'' - z^2$.

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Systematic Approach

- Given $y' = f(t, y)$, using order of operations from the inside out, identify functions (and odes that define them) in polynomial form. Build outwards until f is in polynomial form.

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- Apply recurrence relations to find Taylor series.

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\end{aligned}
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7 Cauchy products per term, no transcendental functions!

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Then

$$
\begin{aligned}
y' &= u_4, \\
u_1' &= u_2 u_4, \\
u_2' &= -u_1 u_4, \\
u_3' &= 3u_3 u_1' = 3u_2 u_3 u_4, \\
u_4' &= -u_5 u_3' = -3u_2 u_3 u_4 u_5, \\
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$u_4' = -3u_5 u_7$, $u_5' = 3u_4 u_7$.

5 Cauchy products per term, no transcendental functions.

History

- JMU: Picard iteration $\Rightarrow$ Taylor series for polynomial systems (Parker & Sochacki 1996), at most quadratic, can de-couple, algebraic structure (Carothers *et al.* 2005), A priori error bounds (Warne *et al.* 2006), Power series substitution, minimizing computation, regular singular points, “better” differential equation representation, delay, Chebyshev etc. (2008-current)
- Automatic Differentiation (AD) community (1959-present): Powerful tool for numerically calculating derivatives, extended to Taylor series (80’s), can hide details from user, not as well known as it should be.
- Fehlberg 1964 NASA report: N-body problem, factor of five improvement.
- Kerner (1980) polynomial systems (few examples).
- Holonomic function theory.

Future Directions

- Clear understandable document.
- Implement automatic translator/solver.
- Implement a priori error estimate, investigate order/time step size balance.
- Further theoretical advances (regular singular points, analytic functions, normality of numbers).
- Delay differential equations.
- Effectively symplectic solver.