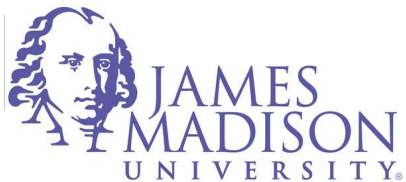


# The Power Series Method for Odes

Stephen Lucas

Department of Mathematics and Statistics  
James Madison University, Harrisonburg VA



# Outline

- The Taylor method and issues.

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- Future Directions.

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The Taylor method is the first taught in a numerical ode class. But even simple right hand sides require lots of work, so Runge-Kutta, multistep etc.

# Taylor Method Example

Van der Pol equation:  $x' = y$ ,  $y' = -x + y - x^2y$ ,  $x(0) = x_0$ ,  $y_0 = y_0$ .

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Ouch! And don't even think about  $x' = \cos(1 + e^{3 \sin x})$ .

# Another Ode Solution Technique

A first course in differential equations introduces a power series substitution method for second order linear differential equations of the form  $p(x)y'' + q(x)y' + r(x)y = f(x)$ , as long as  $p, q, r, f$  are sufficiently simple.

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Almost never seen again, especially when an ode is nonlinear. But, there is no reason why it can't be applied to other odes...

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$$y' = -x + y - x^2y \text{ becomes}$$

$$\sum_{i=1}^{\infty} i y_i t^{i-1} = - \sum_{i=0}^{\infty} x_i t^i + \sum_{i=0}^{\infty} y_i t^i - \left( \sum_{i=0}^{\infty} \left( \sum_{j=0}^i x_j x_{i-j} \right) t^i \right) \left( \sum_{i=0}^{\infty} y_i t^i \right),$$

and equate coefficients.

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Great for polynomial right hand side, but isn't that a major restriction?

# A Bold Claim

- **Any** system of odes with analytic solutions, or analytic functions that can be represented as solutions of odes, can be reformulated in polynomial form – mostly as  $y' = f(t, y)$ , sometimes as  $zy' = f(t, y)$  where  $z$  is a vector of ones or  $t$ 's (regular singular points).

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- A systematic approach can be applied to make the conversion to polynomial form, and identify intermediate variables.

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- Extensive theoretical background (Ed Parker, Dave Carothers).

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If  $y' = \sin z$ , let  $u_1 = \sin z$  and  $u_2 = \cos z$ , then  $y' = u_1$ ,  $u_1' = u_2 z'$  and  $u_2' = -u_1 z'$ .

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If  $y = \log(1 + t)$ , let  $z = 1/(1 + t)$ , then  $y' = z$  and  $z' = -z^2$ .

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- Apply recurrence relations to find Taylor series.

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# History

- JMU: Picard iteration  $\Rightarrow$  Taylor series for polynomial systems (Parker & Sochacki 1996), at most quadratic, can de-couple, algebraic structure (Carothers *et al.* 2005), A priori error bounds (Warne *et al.* 2006), Power series substitution, minimizing computation, regular singular points, “better” differential equation representation, delay, Chebyshev etc. (2008-current)
- Automatic Differentiation (AD) community (1959-present): Powerful tool for numerically calculating derivatives, extended to Taylor series (80’s), can hide details from user, not as well known as it should be.
- Fehlberg 1964 NASA report: N-body problem, factor of five improvement.
- Kerner (1980) polynomial systems (few examples).
- Holonomic function theory.

# Future Directions

- Clear understandable document.
- Implement automatic translator/solver.
- Implement a priori error estimate, investigate order/time step size balance.
- Further theoretical advances (regular singular points, analytic functions, normality of numbers).
- Delay differential equations.
- Effectively symplectic solver.