There are theorems for planar polynomial ODEs that deal with the number of periodic solutions and limit cycles. One interesting theorem for limit cycles is Dulac's theorem which states that for planar polynomial ODEs there can only be a finite number of limit cycles. (It is interesting to note that the proof of this theorem had an error that was not corrected until 70 years later even though it was the basis for one of Hilbert's questions that is still open: What is the minimum number of limit cycles for a quadratic polynomial planar ODE?)

Lawrence Perko in his text *Differential Equations and Dynamical Systems* shows that this is not true for the planar ODE

$$r' = r^3 \sin \frac{1}{r}$$
$$\theta' = 1,$$

where $r = \sqrt{x^2 + y^2}$ and $\theta = atan \frac{y}{x}$. The equilibrium solutions of this ODE are the origin and the circles (limit cycles) $r = \frac{1}{n\pi}$ for n a natural number. These limit cycles converge to the origin. For n even the limit cycles ($r = \frac{1}{n\pi}$) are stable and for n odd they are unstable.

Now consider

$$r' = r(r-1)(r-2)...(r-n) = f(r)$$
$$\theta' = 1$$

which has n limit cycles. The right hand side is analytic, but not polynomial. If we convert this ODE to rectangular coordinates (x, y) we obtain

$$x' = -y + f(r)\frac{x}{r}$$

$$y' = x + f(r)\frac{y}{r}.$$

If we let $z = r$ and $w = z^{-1}$ in this system then we find that

$$x' = -y + xwf(z)$$
$$y' = x + ywf(z)$$
$$z' = f(z)$$
$$w' = -w^2 f(z).$$

This is a polynomial ODE with n limit cycles, but it is now an ODE in $\mathbb{R}^4$.

Consider

$$r' = \sin r = f(r)$$
$$\theta' = 1$$

Now let $u = f(z) = \sin z$ and $v = \cos z$ then

$$x' = -y + xwf(z) = -y + xwu$$
$$y' = x + ywf(z) = x + ywu$$
$$z' = f(z) = u$$
$$w' = -w^2 f(z) = -uw^2.$$
$$u' = vz' = vf(z) = uv$$
$$v' = -uz' = -uf(z) = -u^2.$$

This is a polynomial ODE in $\mathbb{R}^6$ with limit cycles $r = n\pi$ for all natural numbers n.

Consider

$$r' = \sin \frac{1}{r}$$
$$\theta' = 1$$

We let $u = \sin w$; $v = \cos w$ and obtain

$$x' = -y + xwu$$
$$y' = x + ywu$$
$$z' = u$$
$$w' = -uw^2.$$
$$u' = -uvw^2$$
$$v' = u^2 w^2.$$

Consider

$$r' = e^{-\frac{1}{r^2}} \sin \frac{1}{r}$$

$$\theta' = 1$$

We let $u = \sin w$; $v = \cos w$; $p = e^{-w^2}$. This gives $r' = pu$. We have

$$x' = -y + xwpu$$

$$y' = x + ywpu$$

$$z' = pu$$

$$w' = -puw^2.$$

$$u' = -puvw^2$$

$$v' = pu^2w^2$$

$$p' = 2uw^3p^2.$$