Consider the problem of a dog chasing a rabbit running on a given path. Suppose the dog starts at the origin and that the position of the dog is given by

$$\bar{D} = x_D\mathbf{i} + y_D\mathbf{j} = r_D \cos\theta\mathbf{i} + r_D \sin\theta\mathbf{j}$$

and the position of the rabbit is given by

$$\bar{R} = x_R\mathbf{i} + y_R\mathbf{j} = r_R \cos\theta_R\mathbf{i} + r_R \sin\theta_R\mathbf{j}.$$

We will assume the dog always moves toward the rabbit. That is,

$$\bar{D}'(t) = \alpha(\bar{R} - \bar{D}) = \alpha\ \bar{w},$$

where α is a positive scalar. We will call this the dog-rabbit ODE. This ODE can also be written as

$$\begin{aligned} x_D' &= \alpha(x_R - x_D) \\ y_D' &= \alpha(y_R - y_D). \end{aligned}$$

Since $r_D^2 = x_D^2 + y_D^2$, we also have

$$\begin{aligned} r_D' &= \frac{(x_D x_D' + y_D y_D')}{r_D} \\ r_D' &= \cos\theta x_D' + \sin\theta y_D' \\ r_D' &= \alpha\cos\theta(x_R - x_D) + \alpha\sin\theta(y_R - y_D) \\ r_D' &= -\alpha(x_D\cos\theta + y_D\sin\theta) + \alpha(x_R\cos\theta + y_R\sin\theta) \\ r_D' &= -\alpha r_D + \alpha(x_R\cos\theta + y_R\sin\theta). \end{aligned}$$

Note that

$$\frac{dr_D}{d\theta} = \frac{-\alpha r_D + \alpha(x_R\cos\theta + y_R\sin\theta)}{\frac{d\theta}{dt}}.$$

Consider the speed of the dog. The dog's speed is given by

$$s_D = \frac{ds}{dt} = ||\bar{D}'(t)|| = |\alpha|||\bar{R} - \bar{D}||.$$

Now note that

$$\sec^2\theta\theta' = \frac{y_D' x_D - x_D' y_D}{x_D^2}$$

giving

$$\theta' = \frac{y_D' \cos\theta - x_D' \sin\theta}{r_D}.$$

Simplifying gives

$$\begin{aligned}\theta' &= \alpha \frac{(y_R - y_D)\sin\theta - (x_R - x_D)\cos\theta)}{r_D} \\ &= \alpha \frac{r_D \cos 2\theta + (y_R \sin\theta - x_R \cos\theta)}{r_D}.\end{aligned}$$

From this we obtain the differential equation

$$\frac{d\theta}{d\theta_R} = \frac{\alpha}{\frac{d\theta_R}{dt}} \frac{r_D \cos 2\theta + (y_R \sin\theta - x_R \cos\theta)}{r_D}.$$

This allows us to consider the ratio α/σ where $\sigma = \frac{d\theta_R}{dt}$.

We consider some cases for α. A famous case is the dog's speed is constant. In this case

$$\alpha = \frac{k}{||\bar{R} - \bar{D}||},$$

where k is the constant speed of the dog. The dog-rabbit ODE can be written in the form

$$\bar{D}'(t) = k \frac{\bar{R} - \bar{D}}{||\bar{R} - \bar{D}||} = k\ \bar{v},$$

where $\bar{v}$ is the unit vector in the direction from the dog to the rabbit.

Another interesting case is when α is a constant. In this case the speed of the dog is given by $|\alpha|||\bar{R}-\bar{D}||$. Therefore, the dog moves fast when it is far from the rabbit and the speed of the dog approaches 0 as the dog approaches the rabbit. The dog-rabbit ODEs have the form

$$x_D' = \alpha(x_R - x_D)$$
$$y_D' = \alpha(y_R - y_D)$$

and are linear. Each ODE in this system has the form

$$u' + \alpha u = \alpha u_R.$$

The solution to this equation is

$$u(t) = u(0)e^{-\alpha t} + e^{-\alpha t}\int_0^t \alpha e^{\alpha\tau} u_R(\tau)d\tau.$$

Suppose the rabbit runs in a periodic motion so that

$$x_R(t+T) = x_R(t)\ ;\ y_R(t+\tau) = y_R(t)$$

for all t and fixed period T. We have

$$u(t+T) = u(0)e^{-\alpha(t+T)} + e^{-\alpha(t+T)}\int_0^{t+T} \alpha e^{\alpha\tau} u_R(\tau)d\tau$$

$$u(t+T) - u(t) = u(0)e^{-\alpha t}(e^{-\alpha T}+1) + e^{-\alpha t}\int_{-T}^{t} \alpha e^{\alpha\tau} u_R(\tau)d\tau -$$
$$e^{-\alpha t}\int_0^t \alpha e^{\alpha\tau} u_R(\tau)d\tau$$
$$u(t+T) - u(t) = u(0)e^{-\alpha t}(e^{-\alpha T}+1) + e^{-\alpha t}\int_{-T}^{0} \alpha e^{\alpha\tau} u_R(\tau)d\tau$$

From this last equation, it is seen that

$$\lim_{t\to\infty}(u(t+T) - u(t)) = 0.$$

Since this is true for both x_D and y_D, we see that if the rabbit runs in a periodic path then eventually the dog will also and with the same period.

Since the integral in

$$u(t) = u(0)e^{-\alpha t} + e^{-\alpha t}\int_0^t \alpha e^{\alpha\tau} u_R(\tau)d\tau,$$

is a convolution, if written as

$$u(t) = u(0)e^{-\alpha t} + \int_0^t \alpha e^{-\alpha(t-\tau)} u_R(\tau) d\tau,$$

we apply the Laplace transform and obtain

$$\mathbb{L}\{u(t)\} = \frac{u(0)}{s+\alpha} + \frac{\alpha}{s+\alpha}\mathbb{L}\{u_R(t)\}.$$

Therefore, as $\alpha \to \infty$ we have $u(t) \to u_R(t)$. That is, the dog follows the rabbit closely or the rabbit's path is a limit cycle for the dog's path with the dog following the motion of the rabbit closely. We also note that

$$\int_0^t \alpha e^{-\alpha(t-\tau)} u_R(\tau) d\tau \to \int_0^t \delta(t-\tau) u_R(\tau) d\tau,$$

in the distributional sense as $\alpha \to \infty$.

If the dog-rabbit ODE is written in terms of the unit vector v as

$$\bar{D}'(t) = \beta \; \bar{v}$$

then the positive scalar β is the speed of the dog. For example, if

$$\beta(\bar{R} - \bar{D}) = \beta(\bar{w})\bar{v} = k \frac{||\bar{w}||}{a + ||\bar{w}||} \; \bar{v}$$

for some number k then the speed of the dog is given by

$$\beta(\bar{w}) = k \frac{||\bar{w}||}{a + ||\bar{w}||}.$$

For this speed, when the dog is far from the rabbit the speed of the dog is close to k and when the dog is close to the rabbit the speed of the dog is close to 0. Also, if $a = 0$ then the speed of the dog is the constant case, k. The case $a = 0$ is a more sensitive ODE than the case $a > 0$ in the sense that numerical methods are more stable for the case $a > 0$ than the case $a = 0$. The dog-rabbit ODE in this case is given by

$$\bar{D}'(t) = \beta(\bar{w}) \; \bar{v} = k \frac{||\bar{w}||}{a + ||\bar{w}||} \; \bar{v} = k \frac{||\bar{w}||}{a + ||\bar{w}||} \; \bar{v}.$$

The last inequality shows that the speed of the dog is $k\frac{||\bar{w}||}{a+||\bar{w}||}$ and is k if $a = 0$. The ODE could also be expressed as

$$\bar{D}'(t) = k \frac{1}{a + ||\bar{w}||} \; \bar{w}.$$

In order to have a simpler polynomial projection of the dog-rabbit ODE we choose

$$\beta = k\frac{||\bar{w}||}{\sqrt{a+(x_D-x_R)^2+(y_D-y_R)^2}}.$$

If $a=0$ in this speed for the dog, we have that the dog's speed is the constant k. If $a>0$ then the speed of the dog is close to k when the dog is far from the rabbit and the speed of the dog is close to 0 when the dog is close to the rabbit as with when the speed of the dog is given by $\beta(\bar{w}) = k\frac{||\bar{w}||}{a+||\bar{w}||}$.

It would be interesting to study different types of $\beta(\bar{w})$.