Suppose that an ant is moving on the unit sphere according to the following differential equations

$$\begin{aligned}x' &= p(x,y,z);\ x(0)=a\\ y' &= q(x,y,z)\ ;\ y(0)=b\\ z' &= r(x,y,z)\ ;\ z(0)=c\end{aligned}$$

then $x^2+y^2+z^2=1$ and $xx'+yy'+zz'=0$.

Consider the special case

$$\begin{aligned}x' &= p(x,y);\ x(0)=a\\ y' &= q(x,y)\ ;\ y(0)=b\\ zz' &= -(xx'+yy')\ ;\ z(0)=c\end{aligned}$$

with $c=\pm\sqrt{1-a^2-b^2}$. We can think of this as a planar ODE 'projected' on the unit sphere. The ant is walking around the top half or bottom half of the unit sphere (depending on the sign of c). We let $w=z^{-1}$ in this system and obtain

$$\begin{aligned}x' &= p(x,y);\ x(0)=a\\ y' &= q(x,y)\ ;\ y(0)=b\\ z' &= -w(xp(x,y)+yq(x,y))\ ;\ z(0)=c\\ w' &= w^3(xp(x,y)+yq(x,y))\ ;\ w(0)=\tfrac{1}{c}\end{aligned}\ .$$

The figure below is a plot of the case

$$\begin{aligned}p(x,y) &= x^2-y^2+2xy-x-3.5y+1\\ q(x,y) &= -x^2+y^2+2xy-y+3.5x-1\end{aligned}$$

with $a=0.25, b=0.125$.

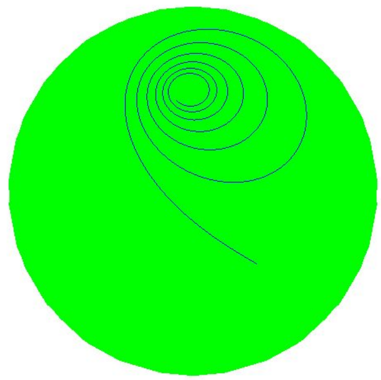

Figure 1. Planar ODE on the Unit Sphere