AN OVERVIEW

THE MODIFIED PICARD METHOD

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ABSTRACT:

This paper discusses how to pose an initial value problem (IVP) ordinary differential equation (ODE)

$$y' = F(t, y); y(t_0) = y_0,$$

where $F : \mathbb{R}^n \to \mathbb{R}^n$ in such a way that a modification of Picard's method will generate the Taylor series solution. We extend the result to the IVP partial differential equation (PDE)

$$u(t, x)_t = F(u, \text{ partial derivatives of } u \text{ in } x) \; ; \; u(t_0, x) = P(x),$$

where $x \epsilon \mathbb{R}^n$,$F : \mathbb{R}^{n+k} \to \mathbb{R}^n$ and $P : \mathbb{R}^m \to \mathbb{R}^n$.

We discuss how to use the method to determine roots of equations using equilibrium solutions of ODE's through inverse functions and Newton's Method. We show that the Maclaurin polynomial can be computed completely algebraically using Picard iteration. Finally, some theorems and questions are asked about the modified Picard method.

Considerations:

After reading this document, you may want to look at the papers on the website

http://educ.jmu.edu/ sochacjs/PSM.html

Another good expository paper after this one is An Adaptive, Highly Accurate and Efficient, Parker-Sochacki Algorithm for Numerical Solutions to Initial Value Ordinary Differential Equation Systems by Jenna Guenther and Morgan Wolf (Paper XIV - Adaptive Time Step Power Series Methods on the website).

I. INTRODUCTION

For continuous F the IVP ODE

$$y'(t) = F(t, y(t))\ ;\ y(t_0) = y_0$$

is equivalent to the integral equation

$$y(t) = y_0 + \int_{t_0}^{t} F(s, y(s))ds.$$

That is, a solution to the IVP ODE is a solution to the integral equation and vice versa.

One can generate a Taylor series solution to the IVP ODE through

$$\begin{aligned}
y_1(t) &= y_0 \\
y_2(t) &= y_0 + y'(t_0)(t - t_0) = y_0 + F(t_0, y_0)(t - t_0) \\
y_3(t) &= y_2(t) + \frac{y''(t_0)}{2}(t - t_0)^2 \\
&\cdot \\
&\cdot \\
&\cdot \\
y_{k+1}(t) &= y_k(t) + \frac{y^{(k)}(t_0)}{k!}(t - t_0)^k.
\end{aligned}$$

Similarly, one can generate a solution to the integral equation using the iterative Picard process

$$\begin{aligned}
p_1(t) &= y_0 \\
p_2(t) &= y_0 + \int_{t_0}^{t} F(s, p_1(s))ds = y_0 + \int_{t_0}^{t} F(s, y_0)ds \\
p_3(t) &= y_0 + \int_{t_0}^{t} F(s, p_2(s))ds \\
&\cdot \\
&\cdot \\
&\cdot \\
p_{k+1}(t) &= y_0 + \int_{t_0}^{t} F(s, p_k(s))ds.
\end{aligned}$$

Lets apply these two processes to a few ODE's and see what the similarities and differences are.

Example 1.

$$y' = y \ ; \ y(0) = 1$$

We see that $y'' = y', y''' = y'', ...$ Therefore, we have $y(0) = 1, y'(0) = y(0) = 1, y''(0) = y'(0) = 1, y'''(0) = y''(0) = 1, ...$ Therefore, the Maclaurin expansion gives

$$y(t) = 1 + t + \frac{t^2}{2} + \frac{t^3}{3!} + ... = e^t.$$

The Picard method for the integral equation gives

$$p_1(t) = 1$$
$$p_2(t) = 1 + \int_0^t p_1(s)ds = \int_0^t 1ds = 1 + t$$
$$p_3(t) = 1 + \int_0^t p_2(s)ds = 1 + \int_0^t (1+s)ds = 1 + t + \frac{t^2}{2}$$
$$p_4(t) = 1 + \int_0^t p_3(s)ds = 1 + \int_0^t (1 + s + \frac{s^2}{2})ds = 1 + t + \frac{t^2}{2} + \frac{t^3}{3!}.$$

It is easily seen that the Picard process produces the Maclaurin series for e^t.

Example 2.

$$y' = ty \ ; \ y(-1) = 1$$

We see that $y'' = y + ty', y''' = y' + y' + ty'', ...$ Therefore, we have $y(-1) = 1, y'(-1) = (-1)y(-1) = -1, y''(-1) = y(-1) + (-1)y'(-1) = 2, y'''(-1) = 2y'(-1) + (-1)y''(-1) = -4, ...$ Therefore, the Taylor expansion gives

$$y(t) = 1 - (t+1) + (t+1)^2 - 2\frac{(t+1)^3}{3} + ... = e^{\frac{(t^2-1)}{2}}.$$

The Picard method for the integral equation gives

$$p_1(t) = 1$$
$$p_2(t) = 1 + \int_{-1}^t sp_1(s)ds = 1 + \int_{-1}^t sds = \frac{1}{2} + \frac{t^2}{2}$$
$$p_3(t) = 1 + \int_{-1}^t sp_2(s)ds = 1 + \int_{-1}^t s(\frac{1}{2} + \frac{s^2}{2})ds = \frac{5}{8} + \frac{t^2}{4} + \frac{t^4}{8}$$

It is not easy to determine a pattern for $p_k(t)$, but we do know it converges to $e^{\frac{(t^2-1)}{2}}$.

Example 3.

$$y_1' = ty_2 \ ; \ y_1(0) = 1$$
$$y_2' = y_1^2 - y_1y_2 \ ; \ y_2(0) = 0$$

In this example we have to calculate Maclaurin series for y_1 and y_2. We determine that $y_1'' = y_2 + ty_2', y_1''' = y_2' + y_2' + ty_2'', ...$ and that $y_2'' = 2y_1y_1' - y_1'y_2 - y_1y_2', y_2''' = 2y_1'^2 + 2y_1y_1'' - y_1''y_2 - y_1'y_2' - y_1'y_2' - y_1y_2'', ...$ We see that in order to get $y_1'''(0)$ we need to first calculate $y_2''(0)$ and similarly for higher derivatives. We determine that $y_1(0) = 1, y_2(0) = 0, y_1'(0) = 0, y_2'(0) = 1, y_1''(0) = 0, y_2''(0) = -1, y_1'''(0) = 2, y_2'''(0) = 1, ...$ This gives

$$y_1(t) = 1 + \frac{t^3}{3} + ...$$

$$y_2(t) = t - \frac{t^2}{2} + \frac{t^3}{6} + ...$$

For the Picard iterates we integrate each equation during each iteration. This gives

$$p_{1,1}(t) = y_1(0) = 1 \ , \qquad p_{2,1}(t) = y_2(0) = 0$$

$$p_{1,2}(t) = 1 + \int_0^t s p_{2,1}(s) ds = 1 + \int_0^t 0 ds = 1$$

$$p_{2,2}(t) = \int_0^t p_{1,1}(s)^2 - p_{1,1}(s)p_{2,1}(s) ds = \int_0^t 1 ds = t$$

$$p_{1,3}(t) = 1 + \int_0^t s p_{2,2}(s) ds = 1 + \int_0^t s^2 ds = 1 + \frac{t^3}{3}$$

$$p_{2,3}(t) = \int_0^t p_{1,2}(s)^2 - p_{1,2}(s)p_{2,2}(s) ds = \int_0^t (1-s) ds = t - \frac{t^2}{2}.$$

We see that we are generating a similar series to the Taylor series.

Example 4.

$$y'' + y = 0 \ ; \ y(0) = 1 \ ; \ y'(0) = -1$$

We make the substitution $y_2 = y'$. Then $y_2(0) = y'(0) = -1$ and we get the following IVP ODE

$$y' = y_2 \ ; \ y(0) = 1$$

$$y_2' = -y \ ; \ y_2(0) = -1.$$

We solve this system in the same manner as in Example 3.

Example 5.

$$y' = \cos(y) + \sin(t) \ ; \ y(1) = 0$$

We find that $y'' = -\sin(y)y' + \cos(t), y''' = -\cos(y)y'^2 - \sin(y)y'' - \sin(t), ...$ This gives

$$y(t) = (1 + \sin(1))(t-1) + \frac{\cos(1)}{2}(t-1)^2 - \frac{(1+\sin(1))^2 + sin(1)}{6}(t-1)^3 + ...$$

for the Taylor series expansion. For the Picard iterates we find that

$$p_1(t) = 0$$

$$p_2(t) = \int_1^t (\cos(p_1(s)) + \sin(s))ds = t - \cos(t) - (1 - \cos(1))$$

$$p_3(t) = \int_1^t (\cos(p_2(s)) + \sin(s))ds = \int_1^t (\cos(s - \cos(s) - (1 - \cos(1))) + \sin(s))ds.$$

This last integral does not have an elementary closed form.

In Examples 1 and 3 we see that the Taylor polynomials and Picard iterates are similar. In Example 2 the results must be different because the initial condition is not at 0. In Example 5 the results must be different because of the sine and cosine. We now show how to repose these examples so that the Taylor and Picard process give similar results.

Let's take a second look at the examples that were not easily solved using Picard iteration. Let's redo Example 2 with a change of the independent variable that transforms the initial conditions to 0.

Example 2. (Revisited)

$$y' = ty \ ; \ y(-1) = 1$$

Let $\tau = t + 1$ and $v(\tau) = y(\tau - 1)$ then $v'(\tau) = y'(\tau - 1) = (\tau - 1)y(\tau - 1) = (\tau - 1)v(\tau)$ and $v(0) = y(-1) = 1$. Now, let $u(\tau) = \tau - 1$ then $u'(\tau) = 1$ and $u(0) = -1$. This gives us the equivalent IVP ODE

$$u' = 1 \ ; \ u(0) = -1$$

$$v' = uv \ ; \ v(0) = 1$$

Note that τ does not explicitly appear on the right hand side (RHS). Also note that the initial conditions are at 0 and that $y(t) = v(t + 1)$.

The Picard method gives

$$q_{1,1}(\tau) = -1 \ ; \ q_{2,1}(\tau) = 1$$

$$q_{1,2}(\tau) = -1 + \int_0^\tau 1ds = -1 + \tau$$

$$q_{2,2}(\tau) = 1 + \int_0^\tau q_{1,1}(s)q_{2,1}(s)ds = 1 + \int_0^\tau -1ds = 1 - \tau$$

$$q_{2,3}(\tau) = 1 + \int_0^\tau q_{1,2}(s)q_{2,2}(s)ds = 1 + \int_0^\tau (-1 + s)(1 - s)ds = 1 - \tau + \tau^2 - \frac{\tau^3}{3}$$

(Note that $q_{1,k}(\tau) = q_{1,2}(\tau)$ for $k \geq 2$.) We now have that

$$p_{1,1}(t) = q_{1,1}(t + 1) = -1$$

$$p_{1,k}(t) = q_{1,k}(t + 1) = t \ ; \ k \geq 2$$

$$p_{2,1}(t) = q_{2,1}(t + 1) = 1$$

$$p_{2,2}(t) = q_{2,2}(t + 1) = 1 - (t + 1)$$

$$p_{2,3}(t) = q_{2,3}(t+1) = 1 - (t+1) + (t+1)^2 - \frac{(t+1)^3}{3}.$$

We see that $p_{2,k}(t)$ is a polynomial that is similar to the Taylor polynomials for Example 2.

We now reconsider Example 5.

Example 5. (Revisited)

$$y' = \cos(y) + \sin(t) \ ; \ y(1) = 0$$

We first let $\tau = t-1, u_1(\tau) = \tau+1$ and $u_2(\tau) = y(\tau+1)$. We then let $u_3(\tau) = \cos(y(\tau+1)), u_4(\tau) = \sin(y(\tau+1)), u_5(\tau) = \sin(\tau+1)$ and $u_6(\tau) = \cos(\tau+1)$. Substituting these into Example 5 gives the equivalent IVP ODE

$$\begin{gathered}
u_1' = 1 \ ; \ u_1(0) = 1 \\
u_2' = u_3 + u_5 \ ; \ u_2(0) = y(1) = 0 \\
u_3' = -\sin(y(\tau+1))u_2' = -u_4(u_3+u_5) \ ; \ u_3(0) = \cos(y(1)) = 1 \\
u_4' = \cos(y(\tau+1))u_2' = u_3(u_3+u_5) \ ; \ u_4(0) = \sin(y(1)) = 0 \\
u_5' = \cos(\tau+1) = u_6 \ ; \ u_5(0) = \sin(1) \\
u_6' = -\sin(\tau+1) = -u_5 \ ; \ u_6(0) = \cos(1)
\end{gathered}$$

The initial conditions are at 0 and the RHS is a polynomial in the dependent variables and autonomous. We leave it as an exercise for you to generate the 6 Picard iterates $p_{j,k}(\tau), j = 1,..,6$ and show that these polynomials are similar to the Taylor polynomials for Example 5 if we replace τ by $t-1$. The polynomials approximating the solution are then given by $p_{1,k}(t-1)$.

Examples 1-5 and the revisits to them lead one to conjecture and prove the following theorem.

Theorem 1

Let $F = (F_1, ..., F_n) : \mathbb{R}^n \to \mathbb{R}^n$ be a polynomial and $Y = (y_1, ..., y_n) : \mathbb{R} \to \mathbb{R}^n$. Consider the IVP ODE

$$y_j' = F_k(Y) \ ; \ y_j(0) = \alpha_j \ ; \ j = 1, ..., n$$

and the Picard iterates

$$p_{j,1}(t) = \alpha_j \ ; \ j = 1, ..., n$$

$$p_{j,k+1}(t) = \alpha_j + \int_0^t F_j(P_k(s))ds \ ; \ k = 1, 2, ... \ ; \ j = 1, ..., n$$

then $p_{j,k+1}$ is the kth Maclaurin Polynomial for y_j plus a polynomial all of whose terms have degree greater than k. (Here $P_k(s) = (p_{1,k}(s), ..., p_{n,k}(s))$.)

The autonomous restriction in Theorem 1, as Example 2 shows, is easily handled in applications.

We now consider an important IVP ODE Theorem 1 can be used on. Isaac Newton posed the following system of differential equations to model the interactive motion of N heavenly bodies.

$$x_i''(t) = \sum_{j \neq i} \frac{m_j(x_j - x_i)}{r_{i,j}^{\frac{3}{2}}}$$

$$y_i''(t) = \sum_{j \neq i} \frac{m_j(y_j - y_i)}{r_{i,j}^{\frac{3}{2}}}$$

$$z_i''(t) = \sum_{j \neq i} \frac{m_j(z_j - z_i)}{r_{i,j}^{\frac{3}{2}}}$$

where $r_{i,j} = [(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2], j = 1, ..., N.$
If we let

$$s_{i,j} = r_{i,j}^{-\frac{1}{2}}$$

this system for the N-body problem can be posed as

$$x_i' = u_i$$

$$y_i' = v_i$$

$$z_i' = w_i$$

$$u_i' = \sum_{j \neq i} m_j(x_j - x_i)s_{i,j}^3$$

$$v_i' = \sum_{j \neq i} m_j(y_j - y_i)s_{i,j}^3$$

$$w_i' = \sum_{j \neq i} m_j(z_j - z_i)s_{i,j}^3$$

$$s_{i,j}' = -\frac{1}{2}s_{i,j}^3[2(x_i - x_j)(u_i - u_j) + 2(y_i - y_j)(v_i - v_j) + 2(z_i - z_j)(w_i - w_j)], i = 1, .., N.$$

We now consider an example for PDE's.

Example 6. Consider the Sine-Gordon IVP PDE

$$u_{tt} = u_{xx} + \sin(u) \ ; \ u(0,x) = e^{(x-x_0)^2} \ , \ u_t(0,x) = 0.$$

Let $v = u_t$. Then $v(0,x) = 0$. Let $w = \sin(u)$ and $z = \cos(u)$ we then have

$$u_t = v \ ; \ u(0,x) = e^{(x-x_0)^2}$$

$$v_t = u_{xx} + w \ ; \ v(0,x) = 0$$

$$w_t = zv \ ; \ w(0,x) = \sin(u(0,x)) = \sin(e^{(x-x_0)^2})$$

$$z_t = -wv \ ; \ z(0,x) = \cos(u(0,x)) = \cos(e^{(x-x_0)^2})$$

The Picard iterates are

$$p_{1,1}(t) = e^{(x-x_0)^2}$$

$$p_{2,1}(t) = 0$$

$$p_{3,1}(t) = \sin(e^{(x-x_0)^2})$$

$$p_{4,1}(t) = \cos(e^{(x-x_0)^2})$$

$$p_{1,2}(t) = e^{(x-x_0)^2} + \int_0^t p_{2,1}(s)ds = e^{(x-x_0)^2} + \int_0^t 0ds = e^{(x-x_0)^2}$$

$$p_{2,2}(t) = e^{(x-x_0)^2} + \int_0^t (\frac{\partial}{\partial x} p_{1,1}(s) + p_{3,1}(s))ds =$$

$$e^{(x-x_0)^2} + \int_0^t (\frac{\partial}{\partial x} e^{(x-x_0)^2} + \sin(e^{(x-x_0)^2}))ds =$$

$$e^{(x-x_0)^2} + (\frac{\partial}{\partial x} e^{(x-x_0)^2} + \sin(e^{(x-x_0)^2}))t$$

$$p_{3,2}(t) = e^{(x-x_0)^2} + \int_0^t p_{4,1}(s)p_{2,1}(s)ds = e^{(x-x_0)^2} + \int_0^t \cos(e^{(x-x_0)^2})0ds = e^{(x-x_0)^2}$$

$$p_{4,2}(t) = e^{(x-x_0)^2} + \int_0^t -p_{3,1}(s)p_{2,1}(s)ds = e^{(x-x_0)^2} + \int_0^t \sin(e^{(x-x_0)^2})0ds = e^{(x-x_0)^2}$$

We see the process is very similar to IVP ODE's and we can treat x as a parameter in the ODE case. We can also make the substitution $y = u_x$ giving $y(0,x) = \frac{\partial}{\partial x} e^{(x-x_0)^2}$. This leads to the system

$$u_t = v \ ; \ u(0,x) = e^{(x-x_0)^2}$$

$$v_t = y_x + w \ ; \ v(0,x) = 0$$

$$w_t = zv \ ; \ w(0,x) = \sin(u(0,x)) = \sin(e^{(x-x_0)^2})$$

$$z_t = -wv \ ; \ z(0,x) = \cos(u(0,x)) = \cos(e^{(x-x_0)^2})$$

$$y_t = u_{xt} = v_x \ ; \ y(0,x) = \frac{\partial}{\partial x} e^{(x-x_0)^2}.$$

The Picard iterates for the IVP ODE can be obtained in a manner similar as above.

This example leads to a theorem similar to Theorem 1.

Theorem 2

Let $F = (F_1, ..., F_n) : \mathbb{R}^m \to \mathbb{R}^n$ be a polynomial, $U = (u_1, ..., u_n) : \mathbb{R} \to \mathbb{R}^n$ and $A = (A_1, ..., A_n) : \mathbb{R}^m - > \mathbb{R}^n$. Consider the IVP PDE

$$\frac{\partial u_j}{\partial t}(t, x) = F_j(x, U, x \; partial \; derivatives \; of \; U) \; ; \; u_j(0, x) = A_j(x),$$

and the Picard iterates

$$v_{j,1}(t) = A_j(x), j = 1, .., n$$

$$v_{j,k+1}(t) = A_j(x) + \int_0^t P_j(x, V_k(s), x \; partial \; derivatives \; of \; V_k))ds \; ; \; j = 1, 2, ...$$

then $v_{j,k+1}$ is the kth Maclaurin Polynomial for $u_j(t, x)$ in t with coefficients depending on x plus a polynomial in t with coefficients depending on x all of whose terms have degree greater than k in t. (Here $V_k(s) = (v_{1,k}(s), ..., v_{n,k}(s))$.)

We now show how to generate the Maclaurin polynomials guaranteed by the above theorems algebraically.

II. GENERATING MACLAURIN POLYNOMIALS

Consider the IVP ODE

$$Y' = AY + B \; ; \; y(0) = w$$

where A is an nxn real (constant) matrix, $Y = (y_1, ...y_n)$, $B = (B_1, ..., B_n)$ and $w = (w_1, ..., w_n)$. Now let

$$p_1(t) = w$$

and from Theorem 1 assume that

$$p_{k+1}(t) = \alpha_0 + \alpha_1 t + ... + \alpha_k t^k + \beta_{k+1} t^{k+1} + ... + \beta_m t^{m+1} = \sum_{j=0}^{n} \alpha_j t^j + \sum_{j=k+1}^{m} \beta_j t^j$$

where $\alpha_0 = w$ and $\alpha_1, ..., \alpha_m \epsilon \mathbb{R}^n$ then

$$p_{k+2}(t) = \alpha_0 + \int_0^t (Ap_{k+1}(s) + B)ds$$

$$= \alpha_0 + \int_0^t (A(\sum_{j=0}^{k} \alpha_j s^j + \sum_{j=k+1}^{m} \beta_j s^j + B)ds$$

$$= \alpha_0 + (B + A\alpha_0)t + \frac{1}{2}A\alpha_1 t^2 + ... + \frac{1}{k+1}A\alpha_k t^{k+1} + ...$$

From this we see that

$$\alpha_0 = w$$
$$\alpha_1 = B + A\alpha_0$$
$$\alpha_2 = \frac{1}{2}A\alpha_1$$
$$\cdot$$
$$\cdot$$
$$\cdot$$
$$\alpha_k = \frac{1}{k}A\alpha_{k-1}$$

are the coefficients for the Maclaurin polynomial of degree k for the solution Y. You should apply this technique to Example 3 to get the Maclaurin polynomials for cosine and sine.

Now lets generate the Maclaurin polynomial for the solution of the initial value problem

$$y'(t) = A + By + Cy^2 = Q(y) \;\; y(0) = \alpha.$$

We start the Picard process with

$$P_1(t) = \alpha = M_1(t)$$

and then define

$$P_{k+1}(t) = \alpha + \int_0^t Q(P_k(s))ds = \alpha + \int_0^t (A + BP_k(s) + CP_k(s)^2)ds.$$

This integral equation is equivalent to the IVODE

$$P'_{k+1}(t) = A + BP_k(t) + CP_k(t)^2 = Q(P_k(t)) \;\; y(0) = \alpha.$$

In [1] we showed that

$$P_k(t) = M_k(t) + \sum_{n=k}^{2^k-1} b_n t^n,$$

where

$$M_{k+1}(t) = \sum_{n=0}^{k} a_n t^n$$

is the degree k Maclaurin polynomial for the solution y to the IVODE. One can use the integral equation or differential equation for $P_{k+1}(t)$ to solve for a_k and b_k. Since $a_0 = \alpha$ and $a_1 = Q(\alpha)$, this only needs to be done for $k \geq 2$. Using Cauchy products this leads to

$$
\begin{aligned}
P'_{k+1} &= \sum_{n=1}^{k} n a_n t^{n-1} + \sum_{n=k+1}^{2^{k+1}-1} n b_n t^{n-1} \\
&= \sum_{n=0}^{k-1} (n+1) a_{n+1} t^n + \sum_{n=k+1}^{2^{k+1}-1} n b_n t^{n-1} \\
&= A + B \sum_{n=0}^{2^k-1} b_n t^n + C \sum_{n=0}^{2^k-1} b_n t^n \sum_{n=0}^{2^k-1} b_n t^n \\
&= A + B \sum_{n=0}^{2^k-1} b_n t^n + C \sum_{j=0}^{2^k-1} \sum_{n=0}^{2^k-1} b_j b_n t^{n+j} \\
&= A + B \sum_{n=0}^{2^k-1} b_n t^n + C \sum_{n=0}^{2(2^k-1)} d_n t^n,
\end{aligned}
$$

where $b_n = a_n$ for $n \le k$ and $d_n = \sum_{j=0}^{n} b_j b_{n-j}$ for $0 \le n \le 2^k - 1$ and $d_n = \sum_{j=n-2^k+1}^{2^k-1} b_j b_{n-j}$ for $2^k \le n \le 2(2^k - 1)$. By equating like powers it is straightforward to show that

$$
k a_k = B a_{k-1} + C \sum_{j=0}^{k-1} a_j a_{k-1-j}.
$$

That is,

$$
M_{k+1}(t) = M_k(t) + \frac{(B a_{k-1} + C \sum_{j=0}^{k-1} a_j a_{k-1-j})}{k} t^k.
$$

That is, we can obtain the Maclaurin polynomial algebraically. This is easily extended to any polynomial ODE or system of polynomial ODE's. Therefore, for ODE's with polynomial right hand side the Maclaurin coefficients of the solution can be obtained algebraically with the same algorithm. This algorithm we call the Algebraic-Maclaurin algorithm. The Cauchy result shows that this algorithm gives the analytic solution to the IVODE. One can also generate a formula for the $b_n's$ for $n > k$ using the above results. In a future paper, we exploit this to give convergence rates and error estimates for polynomial differential equations. It is also easy to see that one can modify the above algorithm to work for any polynomial system.

The above examples and theorems lead to many interesting questions about the relationship between Taylor polynomials and Picard polynomials, analytic functions and polynomial IVP ODE's, the symbolic and numerical calculations using Picard's method and the best way to pose an IVP ODE or IVP PDE.

For example, consider the IVP ODE
(I)

$$
x' = \sin x; \ x(0) = \alpha.
$$

We let $x_2 = \sin x$ and $x_3 = \cos x$ to obtain

(II)

$$\begin{aligned} x' &= x_2;\ x(0) = \alpha \\ x_2' &= x_2 x_3;\ x_2(0) = \sin(\alpha) \\ x_3' &= -x_2^2;\ x_3(0) = \cos(\alpha). \end{aligned}$$

Notice that in this system one does not have to use the Maclaurin polynomial for $\sin x$ and that the above algorithm will generate the coefficients of the Maclaurin polynomial for x without knowing the one for sin. We give numerical results for these two systems using fourth order Runge-Kutta and fourth, fifth and sixth order Maclaurin polynomials, but first we show that using the polynomial system it is easy to generate the solution to the original IVODE. However, first we show that turning (I) into a system of polynomials (II) also makes it easy to solve.

Using the equation for x_2' and x_3' from (II) it is seen that $x_2^2 + x_3^2 = 1$ so that $x_3' = x_3^2 - 1$ whose solution is

$$x_3 = \frac{1 - e^{2t+2B}}{1 + e^{2t+2B}}.$$

From this we obtain

$$x_2 = \frac{4e^{2t+2B}}{(1 + e^{2t+2B})^2}$$

and since $x' = x_2$, we finally have

$$x = 2 \arctan e^{t+B}$$

We use this exact solution to compare to our numerical results.

In the table below is the error of the results of a simulation with a time step of 0.0625 using fourth order Runge-Kutta on the systems (I-II) and the fifth order Maclaurin polynomial on (II) using the algorithm above for the initial condition $x(0) = \frac{31\pi}{32}$. The computing times were essentially the same. The errors are in comparison with the exact solution given by Maple using 30 digits of accuracy.

Time	Order	R-K on I	R-K on II	Maclaurin on II
0.125	4	$1.44366E-09$	1.35448E-09	1.21177E-09
	5			7.6911E-12
	6			6.14E-14
2	4	3.55865E-09	3.46205E-09	4.43075E-09
	5			6.71039E-11
	6			1.1714E-12

In these results it is seen that the polynomial system gives the best results and that increasing the degree improves the results significantly.

We now differentiate the system of polynomials (II) for x and obtain

$$\begin{aligned} & x'' = x_2' = x_2\, x_3 \\ & x''' = x_2'\, x_3 + x_2\, x_3' = x_2\, x_3{}^2 = x_2\,(1 - x_2{}^2) - x_2{}^3 \\ & x''' - x' + 2(x')^3 = 0 \end{aligned}$$

This last ODE is a polynomial ODE only in x. That is, we have decoupled the system into a polynomial ODE for the solution x to $x' = \sin x$.

Other interesting examples are

$$y' = y^r \ ; \ y(0) = \alpha,$$

(The bifurcation at $r = 1$ arising in the polynomially equivalent systems is particularly interesting.)

$$y' = \frac{1}{y^3} + t \ ; \ y(0) = \frac{1}{8}$$

and

$$u_t = (u_{xx})^{1/3} \ ; \ u(0, x) = f(x)$$

In the first example, the bifurcation at $r = 1$ arising in the polynomially equivalent systems is particularly interesting.

As mentioned earlier, the above methods and concepts can be used to look at inverses and roots of functions.

III. INVERSES AND ROOTS

We now consider determining the roots of the function $f : \mathbb{R} \to \mathbb{R}$. That is, we want to solve $f(x) = 0$ or $f(x) = t$. First, consider

$$f(x(t)) = t$$

then by the chain rule:

$$f'(x)x'(t) = 1$$

Letting $u = [f'(x)]^{-1}$ gives

$$x' = u$$
$$u' = -u^3 f''(x)$$

Choosing $x(0) = a$ gives the IVP ODE

$$x' = u \ ; \ x(0) = a$$
$$u' = -u^3 f''(x) \ ; \ u(0) = 1/f'(a)$$

for determining f^{-1}, since $x = f^{-1}$. If f is a polynomial we can use Theorem 1.

Example 7. Determine f^{-1} for

$$f(x) = \ln(x)$$

Let $v = \ln(x)$ and $w = 1/x$ then $v' = f'(x)x' = wx'$ and $w' = -w^2 x'$ which leads to the polynomial IVP ODE

$$x' = u \ ; \ x(0) = 1$$

$$u' = u^3 w^2 \ ; \ u(0) = 1/f'(1) = 1$$
$$v' = wu \ ; \ v(0) = 0$$
$$w' = -w^2 u \ ; \ w(0) = 1$$

for determining e^x. This example hints at a method for determining a Taylor series expansion for the local inverse of a polynomial.

Newton's method for solving

$$f(x) = 0$$

is "equivalent" to determining the fixed points of

$$x - \frac{f(x)}{f'(x)}.$$

We note that the equilibrium solutions of

$$x' = f(x)$$

and

$$x' = -\frac{f(x)}{f'(x)}$$

are the roots of f if f' is not 0 at the roots.

The solution to this last ODE can be expressed as

$$\ln(f(x(t)) - \ln(f(x(0)) = -t$$

or

$$x(t) = f^{-1}(f(x(0)e^{-t}))$$

so it is easy to see that x approaches $f^{-1}(0)$ a root of f.

Let us now show that the equilibrium solutions of this ODE are stable. Consider

$$x' = -\alpha \frac{f(x)}{f'(x)} = g(x).$$

Then

$$g'(x) = -\alpha(1 - \frac{f(x)f''(x)}{[f'(x)]^2}).$$

Therefore, if x is near a root of f we have $g'(x) < 0$.

If we let $u = [f'(x)]^{-1}$ then we obtain the IVP ODE

$$x' = -\alpha u f(x)$$
$$u' = -u^2 f''(x) x' = -u^2 f''(x)(-\alpha u f(x)) = \alpha u^3 f(x) f''(x).$$

If we pick a value for $x(0)$ that is close to a root a of f then the solution to

$$x' = -\alpha u f(x)\ ;\ x(0) = a$$
$$u' = -u^2 f''(x)x' = -u^2 f''(x)(-\alpha u f(x)) = \alpha u^3 f(x) f''(x)\ ;\ u(0) = 1/f'(a).$$

approaches the root a of f. If f is a polynomial we can use Theorem 1 to determine the roots of f. If f is not a polynomial we make a projection. The next example shows the system of ODE's that can be used to determine the root of a non-polynomial f.

Example 8.

$$f(x) = x\cos(x) + e^{-x}$$

Let $v = \cos(x)$,$w = \sin(x)$ and $z = e^{-x}$ then from the preceding discussion

$$x' = -\alpha u(xv + z) = y$$
$$u' = \alpha u^3(xv + z)(-2w - xv + z) = -u^2 y(-2w - xv + z)$$
$$v' = -w(-\alpha u(xv + z)) = -wy$$
$$w' = v(-\alpha u(xv + z)) = vy$$
$$z' = -z(-\alpha u(xv + z)) = -zy$$
$$y' = -\alpha(-u^2 y f''(x)f(x) + uf'(x)y) = -uy^2(-2w - xv + z) - \alpha u(yv - xwy - zy)$$

Now consider the ODE

$$x'(t) = -\alpha(x)f(x).$$

One can determine $\alpha(x)$ so that the roots of f are stable equilibria for this ODE. For example, try $\alpha(x) = D(f(x)) = f'(x)$ or $\alpha(x) = D(f(x))^{-1} = [f'(x)]^{-1}$.

For another ODE, consider

$$\frac{d}{dt}f(x(t)) = -\alpha t^k f(x)$$

where α is a constant. By adjusting α and k during the calculation of the solution to the ODE we can speed up the convergence to the root of f.

This application of Newton's method is easily extendable to $f : \mathbb{R}^n \to \mathbb{R}^n$ by using the last equation.

Other Search Ideas

Consider the problem of determining the zeroes of $F : \mathbb{R}^n \to \mathbb{R}^n$. Newton's method is

$$x_{k+1} = x_k - [DF(x_k)]^{-1}F(x_k)$$

The corresponding ODE is

$$x' = -[DF(x)]^{-1}F(x).$$

Consider

$$(F(x(t))' = DF(x)x'$$

and the problem of determining x so that

$$DF(x)x' = g$$

for a given g. If we replace x' by h we have

$$DF \circ h = g$$

or

$$h = [DF]^{-1}g.$$

Now consider the ODE

$$x' = -[DF(x)]^T F(x)$$

The equilibrium solutions of this ODE are the zeroes of F. Note that

$$\begin{aligned}\frac{d}{dt}||F(x)||^2 &= \frac{d}{dt} < F(x), F(x) > \\ &= -2||[DF(x)]^T F(x)||^2.\end{aligned}$$

If $G = -[DF]^T F$ then

$$DG(x) = -D([DF]^T F) = -[DF]^T DF - [D_1([DF]^T)F...D_n([DF]^T)F]$$

so that

$$DG(z) = -[DF(z)]^T DF(z)$$

at a zero z of F. Note that $DG(z)$ is a negative definite matrix.

Now consider the generalization of this last ODE as

$$x'(t) = L \circ F(x)$$

and

$$\frac{d}{dt} < F(x(t)), F(x(t)) >=$$

$$< DF(x)x'(t), F(x) > + < F(x), DF(x)x'(t) >=$$

$$< DF \circ L \circ F, F > + < F, DF \circ L \circ F > .$$

Choose $L = -g(x)DF^T$ where $g : \mathbb{R}^n \to \mathbb{R} > 0$.

The method of steepest descent

$$x_{k+1} = x_k + \alpha_k \nabla f(x_k)$$

can be studied by looking at the ODE

$$x' = \alpha \nabla f(x).$$

IV. PROPERTIES OF PROJECTIVELY POLYNOMIAL ODE'S

The examples and theorems presented above lead to the following definition.

Definition

Let $x : \mathbb{R} \to \mathbb{R}$ then x is said to be projectively polynomial if there is an $n \epsilon N$ and $\alpha_2, ..., \alpha_n \epsilon \mathbb{R}$, $x_2, ..., x_n : \mathbb{R} \to \mathbb{R}$ and a polynomial $Q : \mathbb{R}^n \to \mathbb{R}^n$ so that $y = (x, x_2, ..., x_n)^T$ satisfies

$$y' = Q(y) \ ; \ y(0) = \begin{pmatrix} x(0) \\ \alpha_2 \\ . \\ . \\ . \\ \\ \alpha_n \end{pmatrix}$$

If the degree of $Q = k$ then we write $x \epsilon P_{n,k}$. We let $P_k = \cup_n P_{n,k}$ and $P = \cup_k P_k$.

We also define the set of analytic functions by

$$A = \{f | f : \mathbb{R} \to \mathbb{R} \text{ analytic}\}.$$

NOTES

[1] $P_m \subset P_k$ for $k > m$.

[2] If $f \epsilon P$ then $f' \epsilon P$

[3] If $f, g \epsilon P$ then $f + g, fg$ and $f \circ g \epsilon P$.

[4] If $f \epsilon P$ and $f(0) \neq 0$ then $\frac{1}{f} \epsilon P$

[5] If $f \epsilon P$ and $f'(0) \neq 0$ then $f^{-1} \epsilon P$

[6] If $f \epsilon P$ and $y' = f(y)$ then $y \epsilon P$

[7] $P_2 = P$.

[8] $P \subset A$. (HARD!)

[9] $P, A/P$ are dense in A.

[10] $P_1 = \text{span } \{t^n e^{\alpha t} | n \epsilon \{0, 1, 2, ...\}, \alpha \epsilon C\}$
$(x' = Mx \quad ; \qquad M$ an n by n matrix).

[11] $u \epsilon P$ if and only if there is a polynomial $Q : \mathbb{R}^n \to \mathbb{R}^n$ so that $Q(u, u', ..., u^n) = 0$. (Grobner Bases)

[12] $\tan t \epsilon P_2 \qquad (y' = 1 + y^2) \qquad \tan t \in P_1$.
$P_{2,1} = \{y | y' = Ay^2 + By + C\}$, $(\frac{1}{t}, \tan t$ and $\tanh t)$.

[13] What is $P_{2,2}$? Consider $z' = p(z)$, $p : C \to C$ a polynomial of degree 2.

[14] (Pade') $x(t) = \frac{\sum_{i=0}^{N} a_i t^i}{1+\sum_{i=1}^{J} b_i t^i} = \sum_{i=0}^{K} M_i t^i$

REFERENCES

[1] Parker, G. Edgar, and Sochacki, James S., Implementing the Picard Iteration, Neural, Parallel, and Scientific Computation, 4 (1996), 97-112.

[2] Parker, G. Edgar, and Sochacki, James S., A Picard-McLaurin Theorem for Initial Value PDE's, *Abstract Analysis and its Applications*, Vol. 5, No. 1, (2000) 47–63.

[3] Pruett, C.D. Rudmin, J.W. and Lacy, J.M., An adaptive N-body algorithm of optimal order, Journal of Computational Physics, 187 (2003), pp. 298-317.

[4] Liu James, Sochacki James and Paul Dostert, Singular perturbations and approximations for integrodifferential equations, *Proceedings of the International Conference on Control and Differential Equations*, Lecture Notes in Pure and Applied Mathematics, Differential Equations and Control Theory, Vol. 225, (2002).

[5] Liu James, Parker Edgar, Sochacki James and Knutsen Aren, Approximation methods for integrodifferential equations, *Proceedings of the International Conference on Dynamical Systems and Applications,III*,383–390, (2001).