HOMEWORK 4 **DUE: Fri., APR. 20** **NAME:**

DIRECTIONS:

- Turn in your homework as **SINGLE-SIDED** typed or handwritten pages.
- **STAPLE** your homework together. Do not use paper clips, folds, etc.
- **STAPLE** this page to the front of your homework.
- Be sure to write your name on your homework.
- Show all work, **clearly and in order**.

You will lose point 0.5 points if one or more of these instructions are not followed.

Questions	Points	Score
1	2	
2	2	
3	2	
4	2	
5	2	
Extra Credit 1	2	
Extra Credit 2	1	
Extra Credit 3	2	
Extra Credit 4	1	
Total	10	

Problem 1: (2 points) Show that the composition of bijections is a bijection.

Proof: let $f : A \to B$ and $g : B \to C$ be bijections. Simply show that $h = g \circ f$ is also one to one and onto.

Problem 2: (2 points) If A is finite and $x \notin A$, then $A \cup \{x\}$ is finite and $Card\,(A \cup \{x\}) = Card\,(A) + 1$.

Proof: A is finite means that $A \approx \mathbb{N}_k$ and $Card(A) = k$. Define a function $f : A \cup \{x\} \to \mathbb{N}_k$ such that

$$f(j) = a_j, \quad j = 1, 2, 3, ..., k,$$

and

$$f(k+1) = x.$$

Clearly since $x \notin A$, this is a bijection to $\mathbb{N}_{k+1}$ so $A \cup \{x\}$ is finite and $Card\,(A \cup \{x\}) = k + 1$. Q.E.D.

Problem 3: (2 points) If B is a finite set and $A \subseteq B$ then A is finite and $Card\,(A) \leq Card\,(B)$. *Hint: Use induction and problem 2.*

Proof: B finite means that $B \approx \mathbb{N}_k$ for some $k \in \mathbb{N}$ and $Card(B) = k$.

The minimal case: Consider a set $D = \{x\}$ with one element, so $Card(D) = 1$. The subsets of D are $C = \emptyset$ or $C = D$. Then clearly C is finite and $Card(C) = 0, 1 \leq Card(D)$.

The induction hypothesis: Suppose $C \subseteq D$ where $Card(D) = k$ and $Card(C) \leq k$.

Now let $Card(B) = k + 1$ and $A \subseteq B$. If $A = B$ then we are done. If $A \subset B$, then $\exists x \in B \setminus A$ such that $A \subseteq B \setminus \{x\}$ and $Card(B \setminus \{x\}) = k$ by problem (4). Then by our induction hypolthesis A is finite and $Card(A) \leq k$.

Problem 4: (2 points) If A is a subset of a countable set B, then A is countable. *Hint: Use problem 3.*

Proof: If B is countable, then it is either finite or denumerable. If it is finite problem 5 implies that A is also finite and we are done. If B is denumerable, then by the proof from class (subsets of denumerable sets are countable), A is also countable. Q.E.D.

Problem 5: (2 points) So that if D is a denumerable set and $f : D \to A$ is onto, then there is a $g : A \to D$ such that g is 1-1.

Proof: Let D be denumerable then there exists an $h : \mathbb{N} \to D$ which is a bijection (i.e. $h(j) \in D, \forall j \in \mathbb{N}$. Define $g : A \to D$ by $g(a) = h(j)$ where j is the least integer such that $h(j) \in f^{-1}(\{a\})$. So $g(a) \in f^{-1}(\{a\}) \Longrightarrow f(g(a)) = a \forall a \in A$.

Clearly g is 1-1 since if $g(u) = g(v)$ the $u = f(g(u)) = f(g(v)) = v$. Q.E.D.

Extra Credit:

Extra Credit 1: (2 points) Let A and B be sets and let $f : A \to B$ be a function. Suppose that $\{A_i\}_{i \in I}$ is a collection of subsets of A and $\{B_j\}_{j \in J}$ is a collection of subsets of B.

(a) (1 point) Show that $f\,(\cup_{i \in I} A_i) = \cup_{i \in I} f\,(A_i)$.

Proof: Let $y \in f\,(\cup_{i \in I} A_i) \iff \exists x \in U_{i \in I} A_i$ such that $f(x) = y$. By the definition of the union, this is true if and only if $x \in A_i$ for some $i \in I$. Hence $y = f(x) \in f\,(A_i) \iff y \in \cup_{i \in I} f\,(A_i)$. Q.E.D.

(b) (1 point) Suppose f is a bijection. Show that $f^{-1}\left(\cap_{j\in J}B_j\right) = \cap_{j\in J}f^{-1}\left(B_j\right)$.

Proof: Let $y \in f^{-1}\left(\cap_{j\in J}B_j\right) \iff f(x) = y \in \cap_{j\in J}B_j$. By the definition of the intersection of sets, this is true if and only if $y \in B_j \forall j \in J \iff x = f^{-1}(y) \in f^{-1}\left(B_j\right) forall j \in J \iff x \in \cap_{j\in J}f^{-1}\left(B_j\right)$. Q.E.D.

Extra Credit 2: (1 points) For following functions, find $f(A)$ and $f^{-1}(B)$.

(a) (0.5 point) $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = \sin x$, $A = \{-2, -1, 0, 1, 2\}$, $B = \{0, 1, 2\}$.

$$f(A) = \{0, \pm\sin(1), \pm\sin(2)\}, f^{-1}(B) = \{2n\pi | n \in \mathbb{Z}\} \cup \left\{\frac{\pi}{2} + 2n\pi | n \in \mathbb{Z}\right\}$$

(b) (0.5 point) $f : \mathbb{R} \to \mathbb{Z}$ is is the floor function defined by

$$f(x) = \lfloor x \rfloor = n$$

where $n \leq x < n + 1$ for $n \in \mathbb{N}$ and $A = (0, 5)$, $B = \{0, 1, 2\}$.

$$f(A) = \{0, 1, 2, 3, 4\}, f^{-1}(B) = [0, 3).$$

Extra Credit 3 (2 points) Let $f : A \to B$ and $B' \subseteq B$.

(a) (1 point) Prove that $f\left(f^{-1}(B')\right) \subseteq B'$.

Proof: Let $y \in f\left(f^{-1}(B')\right) \Longrightarrow \exists x \in f^{-1}(B')$ such that $f(x) = y \in B$. But the definition of $x \in f^{-1}(B')$ is that there esists a $b \in B'$ such that $f(x) = b \in B'$. That is $y = b = f(x) \in B'$. So we have shown $f\left(f^{-1}(B')\right) \subseteq B'$.

Q.E.D.

(b) (1 point) Prove that if f is onto, then $f\left(f^{-1}(B')\right) = B'$.

Proof: Part (a) implies that $f\left(f^{-1}(B')\right) \subseteq B'$. Now let $y \in B'$. Then there exists an $x \in A$ such that $f(x) = y$ since f is onto. So $x \in F^{-1}(B) \Longrightarrow y = f(x) \in f\left(f^{-1}(B)\right)$. Q.E.D.

Extra Credit 4: (1 point) Prove or find a counterexample to the following conjecture. Assume $f : X \to Y$ and $A, B \subset X$ If $f(A) \setminus f(B) = \emptyset$, then $f(A \setminus B) = \emptyset$.

Counter-examples: Let $X = \{a, b, c, d, ..., z\}$, $A = \{p, q, r\}$, $B = \{p\}$, and $Y = \{\pi\}$. Then let $f : X \to Y$ be denoted by $f(x) = \pi$ for all $x \in X$.