HOMEWORK 5 **DUE: Fri., Apr. 27** **NAME:**

DIRECTIONS:

- Turn in your homework as **SINGLE-SIDED** typed or handwritten pages.
- **STAPLE** your homework together. Do not use paper clips, folds, etc.
- **STAPLE** this page to the front of your homework.
- Be sure to write your name on your homework.
- Show all work, **clearly and in order**.

You will lose point 0.5 points for each instruction not followed.

Questions	Points	Score
1	1	
2	2	
3	2	
4	3	
5	1	
6	1	
Total	10	

Problem 1: (1 point) Let a be a positive rational number. Let $A = \{x \in \mathbb{Q} | x^2 < a\}$. Show that A is bounded in $\mathbb{Q}$. Does it have a least upper bound?

Proof: We must show three things: (1) $A \neq \emptyset$, (2) A has an upper bound, and (3) A has a lower bound.

(1) We know $a > 0$ so we know that either $a > 1$ or $a \leq 1$.

- Suppose $a > 1$. Then $1^2 = 1 < a \Longrightarrow 1 \in A$ so $A \neq \emptyset$.
- Suppose $a \leq 1$. Then since $a > 0 \Longrightarrow a^2 \leq a \Longrightarrow a \in A$ so $A \neq \emptyset$.

(2) Consider only $x \in A$ such that $x \geq 0$. Clearly 0 is an upper bound for the negative numbers so we need only find an upper bound for the positive numbers. Now we want to show that there exists an $M \in \mathbb{Q}$ such that $x \leq M$ for all $x \in A$. Now, we either have $a \in A$ or $a \notin A$.

- Suppose $a \in A \Longrightarrow a^2 < a \Longrightarrow a < 1$ and so for all $x \in A$, $x^2 < a < 1 \Longrightarrow x < 1$. So $M = 1$ is an upper bound.
- Suppose $a \notin A$. Then $a^2 \geq a$, then we have either (i) $a = 1 \Longrightarrow x^2 < 1 \iff x < 1$ so $M = 1$ is an upper bound or (ii) $a > 1 \Longrightarrow x^2 < a < a^2 \in \mathbb{Q}$. Hence $x^2 < a^2 \iff x < a$ so choose $M = a$ as an upper bound.

Problem 2: (2 points) Let $\wp(X)$ be the power set of X. Define the binary relation on $\wp(X)$ as follows: $A, B \in \wp(X)$, $A \sim B \iff A \subseteq B$. Verify that $\wp(X)$ under this relation is a partially ordered set (poset).

Proof: Let $A, B, C \in \wp(X)$. Then clearly

(i) $A \subseteq A \Longrightarrow A \sim A$.

(ii) $A \sim B$ and $B \sim A \Longrightarrow A \subseteq B$ and $B \subseteq A \Longrightarrow A = B$.

(iii) $A \sim B$ and $B \sim C \Longrightarrow A \subseteq B \subseteq C \Longrightarrow A \subseteq C$.

Q.E.D.

Problem 3: (2 points) Prove that $\sqrt{2}$ is not a rational number.

Proof: Suppose $\sqrt{2}$ is a rational number. Then $\sqrt{2} = \frac{a}{b}$ for $a, b \in \mathbb{Z}$ such that $b \neq 0$. Let us suppose that this is in its most reduced form. That is, a and b have no common factors. Then

$$2 = \frac{a^2}{b^2} \iff 2b^2 = a^2,$$

which means that a is even (since its square is even). So $a = 2k$ for some $k \in \mathbb{Z}$. Then

$$b^2 = 2k^2,$$

which implies that b too is even. But this is a contradiction because we assumed that a and b had no common factors.

Problem 4: (3 points) Prove that an ordered field has the least upper bound property if and only if it has the greatest lower bound property.

Proof:

Part 1: Let us assume that our ordered field has the least upper bound property. We want to show that any subset $B \neq \emptyset$ which is bounded below has a greatest lower bound.

$$\text{Let } \; L = \{m \mid m \text{ is a lower bound on } \; B\}.$$

B is bounded below so $L \neq \emptyset$. That is, $\forall m \in L$, we have $m \leq b \; \forall b \in B$. So each $b \in B$ is an upper bound on L. Hence, L is bounded above and by the least upper bound property there exists an $\alpha = lub\,(L)$. *Claim:* $\alpha = glb\,(B)$.

First we show that α is a lower bound of B. If not, there exists a $\gamma \in B$ such that $\gamma < \alpha$. But all elements of B are upper bounds of L and $\gamma \in B$ so γ is an upper bound on L smaller than α, which is a contradiction. Therefore, no such γ exists. So α is, in fact, a lower bound on B.

Now consider β such that $\alpha < \beta$ such that β is also a lower bound (i.e. that α is not the greatest lower bound). But if $\beta > \alpha$ this implies that $\beta \notin B$ which means that β cannot be a lower bound. So $\alpha = glb\,(B)$.

Part 2: Let us assume that our ordered field has the greatest lower bound property. We want to show that any subset $B \neq \emptyset$ which is bounded above has a least upper bound.

$$\text{Let } \; U = \{M \mid M \text{ is an upper bound on } \; B\}.$$

The procedure is the same as for part 1.

Q.E.D.

Problem 5: (1 point) Let $a,\ b \in \mathbb{N}$. We define a number $n \in \mathbb{N}$ to be even if $n = 2k$ for some $k \in \mathbb{N}$. Similarly, we define a number $n \in \mathbb{N}$ to be odd, if $n = 2k + 1$ for some $k \in \mathbb{N}$.

(a) (0.5 points) Prove that if a and b are odd, then $a \cdot b$ is also odd.

Proof: a and b odd means that $a = 2k+1$ and $b = 2l+1$ for some $k, l \in \mathbb{N}$. Hence $a \cdot b = 4kl + 2(k+l) + 1 = 2(2kl + k + l) + 1 = 2m + 1$ where $m = 2kl + k + l \in \mathbb{N}$.

Q.E.D.

(b) (0.5 points) Prove that $a \cdot b$ is even if and only if a is even, b is even, or both are even.

Part I: First suppose $a \cdot b$ is even and suppose neither a nor b are even. Then by (a), we have a contradiction.

Problem 6: (1 point) Let r be a rational number such that $r \neq 0$ and s be an irrational number.

(a) (0.5 points) Prove that $r + s$ is irrational.

Proof: Suppose $r + s$ is rational. Then $r + s = \frac{a}{b}$ for $a, b \in \mathbb{Z}$ where $b \neq 0$ and $r = \frac{p}{q}$ where $p, q \in \mathbb{Z}$ and $p \neq 0 \neq q$. Then

$$s = \frac{a}{b} + \frac{p}{q} = \frac{aq + pb}{bq} \in \mathbb{Q},$$

which implies $s \in \mathbb{Q}$, which is a contradiction. Hence $r + s$ must be irrational.

Q.E.D.

(b) (0.5 points) Prove that $r \cdot s$ is irrational.

Proof: Suppose $r \cdot s$ is rational. That is $r \cdot s = \frac{a}{b}$. Then

$$\frac{p}{q} \cdot s = \frac{a}{b} \Longrightarrow s = \frac{aq}{bp},$$

since $p \neq 0 \neq q$. This implies $s \in \mathbb{Q}$ which is a contradiction. Hence $r \cdot s$ must be irrational.

Q.E.D.