

# Fishing for Answers: Investigating Sustainable Harvesting Rate Models

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## 1 Abstract

The purpose of this report is to determine and propose a model by which an optimal harvesting frequency can be determined to maintain a steady population of Alaskan salmon. In order to do so, we look at the growth rates of the salmon population as well as death rates due to harvesting and other factors. Some mortality factors, however, are age-dependent (for instance, only adult fish have been found to cannibalize young fish), so we consider the overall Alaskan salmon population as two subpopulations of the adult and juvenile fish [3]. We then find a differential equation to model the population that relates the adult population, the juvenile population, and time  $t$ . We consider the periodic nature of harvesting. Every year, fisheries target the adult fish that return to reproduce, for the spawning salmon is the most desirable to human consumers [3]. Because the mating season occurs annually and around the same time, harvesting can be represented by a constant, periodic trigonometric function whose amplitude and frequency represent the harvesting rate limit and the frequency per unit time, respectively [6]. Together, the population model with the oscillatory harvesting component result in a nonlinear differential equation that we solve numerically with Euler's method of approximation.

## 2 Mission Statement

Of major concern to fish harvesters is the sustainability of a population over time. A population of fish can be either overpopulated or underpopulated. Harvesters must take into account these states when determining the frequency of their harvest.

If a fish harvester were to overfish, the fish population would be unable to repopulate for the next season; thus, overharvesting is counterproductive towards the long term goal of renewable fish sales. If a fish harvester were to underfish, the population might exceed its carrying capacity and fish would die at an artificially low age, simply because the available resources would be insufficient to sustain that level of population. This is wasteful in the eyes of the harvesters, for those fish might otherwise have been profitable, and they cannot market fish that die of natural causes [2].

In this report, we strive to create a model by which harvesters can determine the optimal harvesting frequency given the specific parameters of

the fish population being harvested. This model then allows fisheries to optimize profits while minimizing harmful effects on the environment and the fish populations.

### 3 Model Design

In this section of the report we propose a model of a fish population undergoing harvesting. Details of the derivation of the model can be found in **Methods**.

In creating a model for a fish population undergoing harvesting, we need to take into account several factors [5]:

1. Valuable fish are those above a certain age, so we exclude juvenile fish,
2. Salmon spawn once per year,
3. Upon spawning, the adult fish die,
4. Valuable fish must replace their parents in order to maintain a viable population.

The Ricker model, created by W. E. Ricker as an equation to represent fish population more precisely than does the standard logistic model, takes into account each of these factors. It relates the change in the population of fish,  $N$ , to the factors contributing to population change:  $\beta_0$  is the original number of juvenile fish,  $\mu$  is the mortality rate of the fish, and  $-\alpha$  is the rate of cannibalism per adult. The Ricker model is given by [7]:

$$\frac{dN}{dt} = \beta_0 N e^{-\alpha N} - \mu N.$$

The two factors that affect the population of valuable fish are the rate of maturation of juvenile fish into valuable adult fish and the rate of mortality. A general form of the above equation states this fact as an equation where  $\beta$  is the rate of maturation of juvenile fish into valuable fish.

$$\frac{dN}{dt} = \beta N - \mu N$$

We may assume that the fish population is great enough such that the function is a density dependent function (meaning it depends on the existing population) and is constant for the entire adult population,  $N(t)$ .

The rate of maturation is more complicated. Considering the cannibalistic nature of salmon [4] and other factors leading to premature death of the

juvenile fish, the rate of maturation is dependent on the survival of the juvenile fish. Here we make several assumptions [5]:

1. The juvenile period is fixed and constant, and upon passing a certain age, fish are considered mature and valuable,
2. Cannibalism only happens to juvenile fish,
3. Only spawning adults (fish in the last year of life) bear young,
4. The spawning adult population is constant and independent of time.

As can be seen in the derivation of equation under **Methods**, the rate of maturation of the juvenile fish depends on the original number of juvenile fish,  $\beta_0$ , and the exponential factor that comes from the cannibalistic nature of salmon.

We now add the harvesting component to our model. In considering a population undergoing harvesting, we make the claims [6]:

1. Harvesting is periodic,
2. Harvesting does not occur during mating season.

Hence, we model the fish harvesting with an oscillatory sine function. In this equation  $a$  is a harvesting rate limit,  $b$  modifies the frequency of fishing cycles per unit time. Note that the additional 1 is included to insure a positive quantity for the sine term:

$$H(t) = a(1 + \sin bt)$$

Thus, the final factor affecting fish population is this harvesting function, so our model becomes:

$$\frac{dN}{dt} = \beta N e^{-\alpha N} - \mu N - a(1 + \sin bt).$$

## 4 Methods

In this section we provide the mathematical analysis that led to the final model provided in **Model Design**.

Recall the model for fish population without the harvesting component:

$$\frac{dN}{dt} = \beta N - \mu N.$$

We assume the population is great enough such that the mortality rate is constant for the entire fish population. The rate of maturation, however, cannot be granted the same simplicity. We must take into account the fact

that not all juvenile fish will become valuable due to the cannibalistic nature of salmon. What follows is the derivation for the rate of maturation,  $\beta$ .

We let  $N_t$  represent the spawning adult population and  $j(t)$  be the number of juvenile fish at time  $t$ . We call the young produced per spawning adult  $\beta_0$ , therefore  $\beta_0(N_t)$  is the initial population of juveniles such that  $j(0) = \beta_0(N_t)$ . Knowing that  $-\alpha$  is the rate of cannibalism per adult, the rate of decrease of juvenile population is given by

$$\frac{dj}{dt} = -\alpha(N_t)j(t).$$

This differential equation can be solved using separation of variables, with the result being given by

$$j(t) = Ae^{-\alpha N_t}.$$

Using the initial condition,  $j(0) = \beta_0(N_t)$ ,  $A$  is found to be  $A = j(0) = \beta_0(N_t)$  and therefore:

$$j(t) = \beta_0(N_t)e^{-\alpha N_t}.$$

Plugging in this result to the original differential equation we get

$$\frac{dN}{dt} = \beta_0(N_t)e^{-\alpha N_t} - \mu N.$$

We now must consider the oscillatory harvesting component of this model. We call the harvesting function  $H(t)$  and consider it to be oscillatory according to the following equation:

$$H(t) = a(1 + \sin bt)$$

Adjusting the above differential equation we get

$$\frac{dN}{dt} = \beta_0(N_t)e^{-\alpha N_t} - \mu N - a(1 + \sin bt).$$

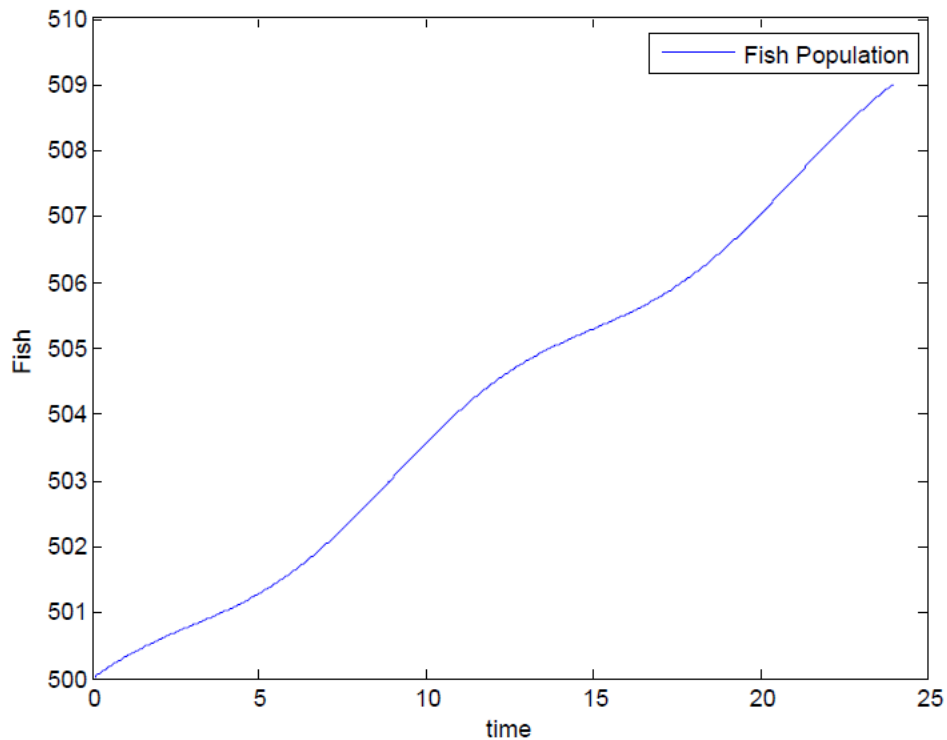
This function is nonlinear, therefore, we solve it using Euler's approximation method. This technique uses tangent lines to the slope, which we know from the differential equation. The general method follows, where  $P' = f(t, P(t))$  [1].

$$\begin{aligned}
 P_1 &= P_0 + f(t_0, P_0)(t_1 - t_0) \\
 &\vdots \\
 P_{n+1} &= P_n + f(t_n, P_n)(t_{n+1} - t_n).
 \end{aligned}$$

In our differential equation, Euler's method takes the form:

$$P_n = P_{n-1} + (\beta_0(N_t)e^{-\alpha N_t} - \mu P_{n-1} - a(1 + \sin bt))(t_n - t_{n-1}).$$

We solve the differential equation numerically using Euler's method on a computer, using a program such as MatLab or Mathematica.



**Figure 1.** This graph shows a plot of fish population over time. The parameters used were:  $\beta_0 = 20$ ,  $\mu = 0.01$ ,  $\alpha = 1.5$ ,  $N_t = 5$ ,  $a = 0.16$ ,  $b = \frac{2\pi}{12}$ .

## 5 Discussion

The numerical result using Euler's equation provides for a slowly increasing curve that has some oscillation in its rate of increase. The increase of fish over time is gradual such that the fish population remains about the same over many harvesting seasons. Using different parameters, however, this result may be different.

TARNADO Environmental Group's goal in writing this report is to create a model that fisheries can use to maintain a steady fish population, not decimating it by overfishing or causing its numbers to skyrocket by under-harvesting. Our model allows fish harvesters to insert the parameters specific to their fish population to determine what frequency of harvesting their population can tolerate. That being said, this model is specific to a generic salmon population and therefore does not fit every case. The purpose of this model is to provide a generalized form by which specific cases can be analyzed given further research into the unique parameters of the system.

Here we address our previous model from the preliminary report on fishing. In that report, we used a logistic model from which we subtracted an oscillatory harvesting component. The equation follows, where  $N$  is population,  $k$  is the rate of growth,  $C$  is the carrying capacity,  $a$  is the harvesting rate limit, and  $b$  is the frequency of fishing cycles per unit time [6]:

$$\frac{dN}{dt} = kN \left( 1 - \frac{N}{C} \right) - a(1 + \sin bt).$$

When solved numerically using Euler's method, the population oscillated just below the carrying capacity.

We adjusted our model to better represent the behavior of salmon. First, we needed a non-logistic model; our initial model assumes constant repopulation over time, but salmon repopulate only once a year during mating season. Thus, rather than assume logistic behavior, we reason through a model for the rate of change of the population of valuable fish, that is, the ones that the fisheries aim to harvest. The resulting equation depends on the development of undesirable juvenile fish into valuable adult fish, the mortality rate of the valuable fish, and the harvesting function. The creation of valuable fish takes into account the cannibalistic nature of salmon, because the cannibalism factors into the death rate of juvenile fish and thereby limits the number that can grow to become valuable. Finally, we retained the harvesting function



based on the Tabakov paper [6], for it models a periodic harvesting season, which describes the fishing of Alaskan salmon.

## 6 Conclusion

This model has harvesting policy implications. Fishing is an important industry in parts of the United States, and yet salmon are currently considered a threatened species [8]. Evidently, the government has not done enough to protect this population, even though it purportedly enforces strict fishing regulations. Our results hint at the complexity of population modeling, thereby suggesting that policy makers must avoid oversimplifying the dynamics of a population and neglecting factors that could irreparably damage a delicate biological system. For example, it has been speculated that climate change will impact salmon's migratory patterns and the timing of their reproductive cycle [8]. As such, climate change should be another factor that the government considers when it creates fishing laws.

## 7 Further Study

This model provides fish harvesters a means of investigating the optimal frequency of harvesting provided their unique parameters. In the future, this model may become more generalizable if research were done about specific areas of the Alaskan coast and the salmon populations that exist there, such as carrying capacity, birth rates, mortality rates, length of spawning season, and other such factors that are geography dependent.

This model could be made to better fit the Alaskan salmon after collecting data on specific areas of the Alaska's ocean geography and its salmon population; with such information, a system of partial differential equations of both space and time could be developed. Adding a spatial component to the model would incorporate salmon's migratory practices.

In the future we could also examine whether this model has a steady state and its behavior over a much greater period of time. On the time scale of the graph provided it seems to increase in an increasing oscillatory fashion. Further studies of this model could calculate the existence of steady states and see whether this model tends towards a function.

## 8 About Us and Acknowledgements

If you give a man a fish, you feed him for a day. If you teach a man how to fish, you feed him for a lifetime. With this in mind, we would like to thank Dr. Eva Strawbridge, not only for her input on this report, but also for the valuable mathematics knowledge that she imparted to our staff. Thank you for teaching us how to fish.

TARNADO, Inc. would also like to call the attention of readers to the hard work of other environmental groups to understand the nuances of the Alaskan salmon and to advise fisheries on how best to preserve this delicate ecosystem. We hope that our report will encourage its readers to learn more about the issue; we urge them to visit our website, upon release of this paper it is still under construction, and to seek out other resources on the matter, because education is the first and most important step to a sustainable existence.

## 9 Author Contribution

This paper represents the product of joint effort: together, the team conceived of the initial idea and brainstormed for it, and all four members reviewed and critiqued the model design. That said, each member contributed his or her particular talents to the project. Research about fish and harvesting and the parameters that were necessary to solve the model numerically was primarily done by Anjuli Uhlig. Trisha Macrae did much of the research into various existing harvesting and population models as well as writing parts of the report. Richard Wang contributed by solving the system using Euler's method, creating the graphs, and rendering the report into its final LaTeX form. No'am Keesom produced the final model using much existing data as well as writing much of the report.

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