A MODEL OF VOLTAGE IN A RESISTOR CIRCUIT AND AN RC CIRCUIT

ARJUN MOORJANI, DANIEL STRAUS, JENNIFER ZELENTY

Abstract. We describe and model the workings of two simple electrical circuits. The circuits modeled are a battery and resistor circuit and an RC (resistor and capacitor) circuit. For the battery and resistor circuit, we derive Ohm's law, which states that the voltage across the resistor is equal to the product of the current running through the resistor and the resistance. For the RC circuit, we derive the exponential decay of voltage supplied by a charged capacitor over time. Additionally, we verify these theoretical relationships with experimental evidence by constructing an actual circuit.

Contents

Date: May 27, 2010.

1. Preliminary Definitions

Before introducing technical definitions in Section 3.1.1, we introduce simple, intuitive definitions [5]. All units used in this paper are SI units.

Definition 1.1 (Charge)**.** Charged matter exhibits electrostatic attraction or repulsion to other charged matter. Charge is either negative or positive; charges of the same sign repel each other and charges of opposite signs have attractive forces between them. The unit of charge is the coulomb C. The smallest charged particle is an electron, and its charge is equal to $1.602 \times 10^{-19}C$; charge is quantized in terms of electrons so it is not possible to have a charge smaller than that of an electron.

Definition 1.2 (Resistor)**.** A resistor opposes the flow of energy, and usually dissipates energy by producing light or heat. Resistance is the measure of how much opposition to current flow a resistor provides. The unit of resistance is an Ohm (Ω).

Definition 1.3 (Capacitor)**.** A capacitor stores charge. Capacitance is the measure of the charge storing ability of a capacitor. The unit for capacitance is a Farad (F).

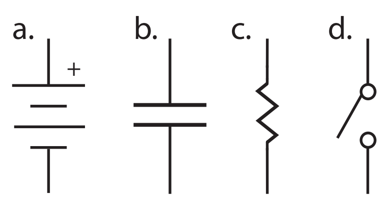

Figure 1. a. A battery. b. A capacitor. c. A resistor. d. A switch.

Definition 1.4 (Voltage)**.** The voltage is equal to energy per unit charge. The unit for voltage is a Volt (V).

Definition 1.5 (Current)**.** The current is equal to the number of charges passing a point per unit time. The unit for current is an Amp (A).

A few additional definitions are required that describe circuits [3].

Definition 1.6 (Series)**.** A series circuit is a circuit in which which the current through each of the components is the same, and the voltage across the components is the sum of the voltages across each component. Components in a series circuit are connected linearly, with one component after another. An example of a series circuit is Figure 2.

Definition 1.7 (Parallel)**.** A parallel circuit is a circuit in which the voltage across each of the components is the same, and the total current is the sum of the currents through each component. In a parallel circuit, the elements are not connected linearly; instead, if there are two resistors, for instance, they are both connected to the same sides of the battery. An example of a parallel circuit is in Figure 6 (when the switch is closed).

2. Statement of Problem

In this report, we analyze two basic electrical circuits: a simple resistor circuit and an RC (resistor and capacitor) circuit. For the resistor circuit, we sought to determine how voltage varies with current. In Section 3.1 we analyze a simple circuit made up of a resistor and battery in series. In this section, we derive Ohm's law and verify the equation $V = IR$ experimentally, thus showing that voltage varies linearly with current.

Fore the RC Circuit, we sought to determine how voltage varies with time (when the capacitor is discharging). In Section 3.3, we analyze a circuit containing a resistor and a charged capacitor in series. We used Kirchhoff's second law to solve a system of differential equations, obtaining Equation 3.13 ($V = V_0 \exp(-t/RC)$). This equation shows that voltage decays exponentially.

2.1. **Importance of Problem.**

These mathematical models are important to consider because they allow us to construct electrical devices. Understanding these principles allows electrical engineers to create products such as toasters, hair dryers, and other heating devices [1]. In addition, resistor circuits are essential for comprehending voltage dividers, which are circuit elements that split voltage between different branches of electrical circuits. They are frequently used in complex electrical circuits.

Capacitors have many practical applications, and in order to use them, we must understand how they function in circuits. For instance, a circuit containing a capacitor is necessary in order to change an AC (alternating current, which has a sinusoidally varying voltage) voltage supply to a DC (constant voltage) voltage supply. All electricity supplied to households is alternating current because when the electrical grid was being designed, alternating current transmission lines allow power to be transmitted over much greater distances [6].[1] To improve

[1]Modern electrical systems use DC to transmit electricity over long distances, but because of the tremendous existing infrastructure, AC current is delivered to houses.

modern circuits, we must fully understand the relationships between the circuit components.

3. Model Design

3.1. **Battery and Resistor Circuit.** The simple circuit, shown in Figure 2, is composed of a battery and resistor in series. The relationship between the current, voltage, and resistance of the system can be expressed by the equation $V = IR$. In this section, we derive this equation using basic definitions of electric field, conductivity, resistivity, resistance, and current density and algebraically solving for the voltage drop across the resistor.

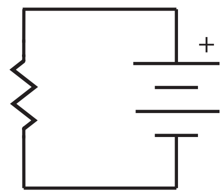

Figure 2. The simple circuit consisting of a battery and a resistor in series.

3.1.1. *Assumptions.* We assume resistance does not vary with temperature [5].

3.1.2. *Ohm's Law.* The following definitons are used in the proof of Ohm's Law.

Definition 3.1 (Electric Field)**.** The electric field E is the force F on a charge q divided by the charge q; in other words, $E = F/q$. Let V be the voltage, and L be the physical length of the resistor (see Figure 3). Then,

$$E = V/L.$$

Definition 3.2 (Conductivity)**.** Let J be the current density, which is defined as the current per unit area of a wire. E is the strength of the electric field. Then,

$$\sigma = J/E,$$

where σ is the current density.

Definition 3.3 (Resistivity)**.**

$$\rho = 1/\sigma,$$

where σ is the conductivity and ρ is the resistivity.

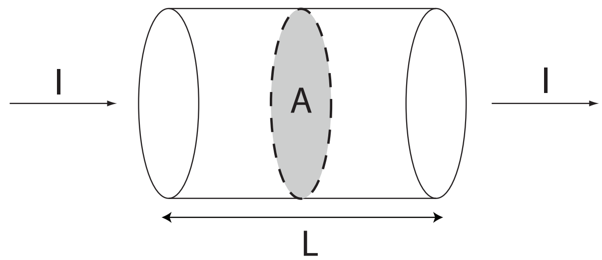

FIGURE 3. A diagram of a resistor. A is the cross-sectional area, L is the length, and I is the current flow.

Definition 3.4 (Resistance)**.** Let R be the resistance, L be the length of the resistor, and A be the cross-sectional area of the resistor. Then,

$$R = L/A.$$

Combining the Definitions 3.2 and 3.3, we obtain that

$$\rho = \frac{EA}{I}. \tag{3.5}$$

Using this result and the Definitions 3.1 and 3.4, we obtain Ohm's law[2], which is

$$V = IR. \tag{3.6}$$

3.2. **Kirchoff's Laws.**

3.2.1. *Kirchoff's Junction Law.* A fundamental principle of electrical circuits is the conservation of charge; no charge can arbitrarily appear in the circuit. The formal statement of this law is a form of the continuity equation, which is

$$\frac{\partial \rho'}{\partial t} + \nabla \cdot J = 0, \tag{3.7}$$

where ρ' is volumetric charge density and J is the current density. The meaning of this equation is that charges are neither created or destroyed; they either flow into or out of a circuit element [4].

Note that the current I in a circuit is equal to the product of the charge density J and the cross-sectional area of the wire.

From the law of conservation of charge, we derive Kirchhoff's law. Consider a junction in a circuit. By the law of conservation of charge, the current going into the junction must equal the current leaving the

[2]Ohm's law fails for a resistor such as a light bulb whose resistance varies with heat.

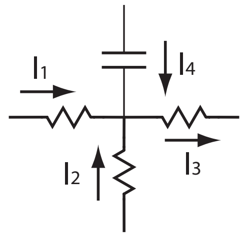

FIGURE 4. In this figure, the currents I_1, I_2, and I_4 flow into the junction, and the current I_3 flows out. By Kirchoff's Junction Law, $I_1 + I_2 + I_4 = I_3$ as all of the current must be conserved at a junction.

junction as otherwise, the continuity equation would be violated. This concept is depicted in Figure 4.

3.2.2. *Kirchhoff's Loop Law.* Kirchoff's Loop Law states that the sum of the voltage changes in any closed loop of a circuit must equal zero. To derive this law, we must first formally define voltage.

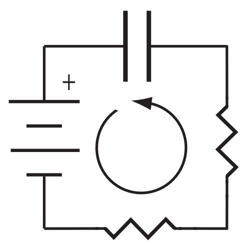

FIGURE 5. By Kirchhoff's Loop law, the sum of the voltage differences across each circuit element must equal zero. In this circuit, the arrow indicates that we would sum the voltages in a counter-clockwise direction, though the orientation of the contour does not matter.

Definition 3.8 (Voltage)**.** Let V be the voltage, and E be the electric field. Then,

$$V = \int_C E \cdot dl.$$

This is a line integral; the contour C can exist in $\mathbb{R}^2$ or $\mathbb{R}^3$. It need not be closed. E can be a function of l, though the definition of voltage is independent of the form of E.

Since electric fields are conservative, the voltage is only dependent on the start and end points of the path; it is independent of the the curve and the parametrization [5]. In other words,

$$\oint_C E \cdot dl = 0. \tag{3.9}$$

Thus, since a circuit is a closed loop, the sum of the voltages of circuit elements in a closed loop is equal to zero.

3.3. **The RC circuit.**

Definition 3.10 (Capacitance)**.** Let C be the capacitance of the capacitor. Then,

$$Q = VC,$$

where Q is the charge on the capacitor and V is the voltage drop across the capacitor[5].

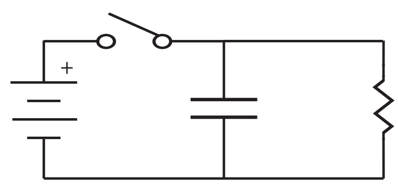

FIGURE 6. This schematic represents the RC circuit discussed in Section 3.3. It is a parallel circuit when the switch is closed, but a series circuit when the switch is open.

Consider the schematic shown in Figure 6. When the switch is closed, the capacitor charges up to $Q = Q_0$. By 3.10, $Q_0 = CV_0$ where V_0 is voltage supplied by the battery. At time $t = 0$, the switch is opened and the capacitor begins to discharge.

We now derive the differential equation for the RC circuit. When the switch is open, By Kirchhoff's Loop Law,

$$Q/C - IR = 0. \tag{3.11}$$

An important relation to note is that $-dQ/dt = I$, which is true because current is simply the amount of charges that pass a certain point per unit time. Then,

$$\frac{dQ}{dt} = -\frac{Q}{RC}. \tag{3.12}$$

Equation 3.12 is a separable differential equation[2]. Note that K is an arbitrary constant.

$$\begin{aligned}\frac{dQ}{dt} &= -\frac{Q}{RC},\\ \frac{dQ}{Q} &= -\frac{dt}{RC},\\ \ln|Q| &= -\frac{t}{RC} + K,\\ Q &= K\exp\left(-\frac{t}{RC}\right).\end{aligned}$$

We are solving an initial value problem because at $t = 0$, $Q = Q_0$, so $K = Q_0$. Thus, the equation describing the decay of charge is

$$Q = Q_0 \exp\left(-\frac{t}{RC}\right).$$

Substituting in $Q = VC$ and $Q_0 = V_0 C$, we arrive at the equation describing voltage, as desired. This equation is

$$V = V_0 \exp\left(-\frac{t}{RC}\right). \tag{3.13}$$

Consider the following theorem:

Theorem 3.14. *Consider the first order, initial value problem* $y' = f(x, y)$, $y(x_0) = y_0$, *and a rectangle,* R, *in the xy-plane such that* $(x_0, y_0) \in R$. *If* f *and* $\frac{\partial f}{\partial y}$ *are continuous on* R, *then there exists an interval* I, *centered at* x_0, *and a unique solution* $y(x)$ *on* I *such that* y *satisfies the above initial value problem* [2].

Our solution in Equation 3.13 is unique because Q and dQ/dt are continuous for all time because of the Law of Conservation of Charge, so regardless of the initial value Q_0, this solution will be unique.

4. Model Implementation

4.1. **Resistor Circuit.**

4.1.1. *Theory.* Since the resistor circuit obeys Ohm's Law ($V = IR$, derived in Section 3.1), the voltage supplied by the battery and the current will have a linear relation and the slope will be the resistance of the resistor. The independent variable is the voltage supplied, and the dependent variable is the current across the resistor.

4.1.2. *Experimental.* We constructed a physical series circuit, depicted in Figure 8, to test Ohm's law. The circuit was composed of a battery and resistor of resistance 9900Ω in series, which means that the positive end of the battery was connected to one end of the resistor while the negative end of the battery was connected to the other end of the resistor. All connections were made using 24 gauge insulated copper wire wire. To take current measurements, an ammeter (a device used to measure current) was connected in series with the resistor so the current that flows through the resistor would be directly measured by the ammeter. In order to measure voltage, a voltmeter (a device used to measure voltage) was placed in parallel with the resistor. The voltmeter was placed in parallel because voltage drops across parallel components of a circuit are identical. Starting at approximately $0V$, the voltage of the DC power supply (which is equivalent to a battery) was increased gradually. The voltage and current were recorded approximately every three volts. These measurements are depicted in Figure 7. The table in Section A.2 contains the experimental data points measured in this circuit.

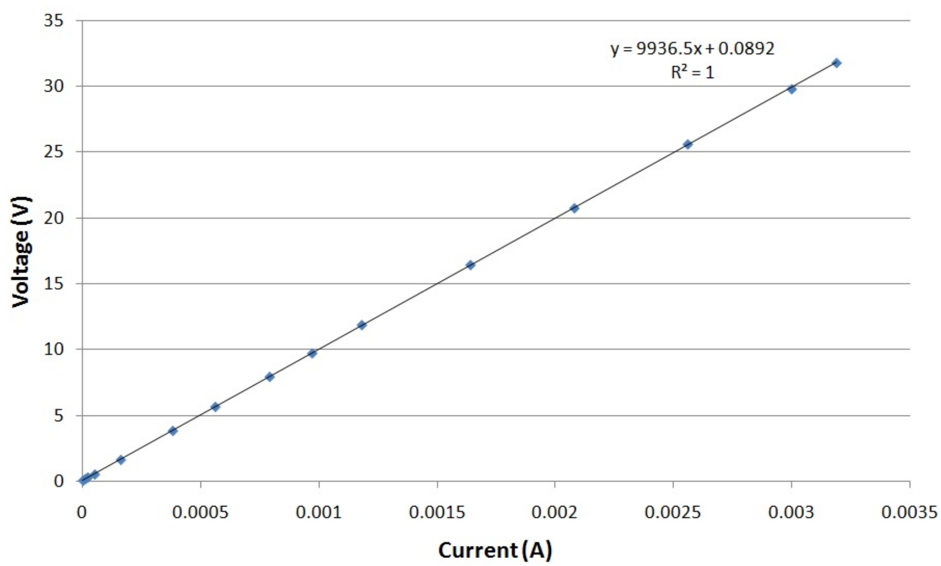

FIGURE 7. This graph plots the voltage versus the current. The slope of the linear fit is 9936Ω. The R^2 value is 1, indicating that the data is well fit by a linear regression.

According to our mathematical model, the graph in Figure 8 should be linear with a slope equal to the resistance. The slope of the graph is

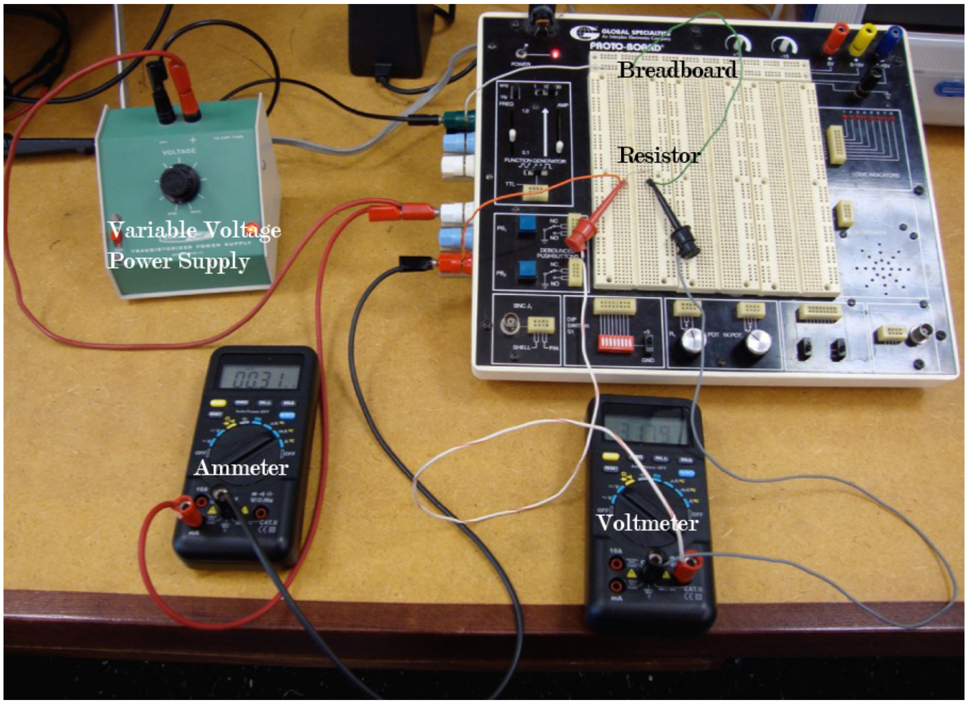

FIGURE 8. The resistor circuit experimental setup. Here, the resistor is in series with a variable voltage power supply. An ammeter is connected in series and a voltmeter is connected in parallel.

9700. Since the actual resistance of the resistor is 9900Ω, these two values are consistent within 2.1%. Therefore, the voltage is proportional to the current in a resistor circuit and the proportionality constant is equal to the resistance. In other words, our results are consistent with the equation $V = IR$.

4.2. **RC Circuit.**

4.2.1. *Theory.* Equation 3.13 allows us to describe the behavior of the RC circuit after the switch is opened and the battery is disconnected from the circuit. As $t \to \infty$, $V \to 0$ because 3.13 is an exponential decay function. The capacitor will discharge until it no longer stores charge. As discussed earlier, the solution is unique because Q and dQ/dt are continuous as a result of conservation of charge.

Furthermore, because charge is quantized, the charge on the capacitor will actually reach zero [5]. It is not possible to have pieces of electrons remaining on the capacitor, so after a sufficiently long time, there will no longer be charges stored in the capacitor.

4.2.2. *Experimental.* We constructed an RC circuit schematically equivalent to Figure 6. In the experimental circuit, a function generator was used as the power supply.[3] A resistor and a capacitor were connected in parallel with the function generator. The function generator was set to output a square wave. The reason a square wave is used is that it alternates between outputting a constant voltage ($4V$ was used in the experiment) and outputting no voltage. This is equivalent to opening and closing a switch in front of a battery or other voltage source. With this setup, when the function generator was outputting zero voltage (the switch is open), the resistor and capacitor were connected in series. When the function generator was supplying a constant voltage, the resistor and capacitor were connected in parallel with the function generator (shown schematically in Figure 6).

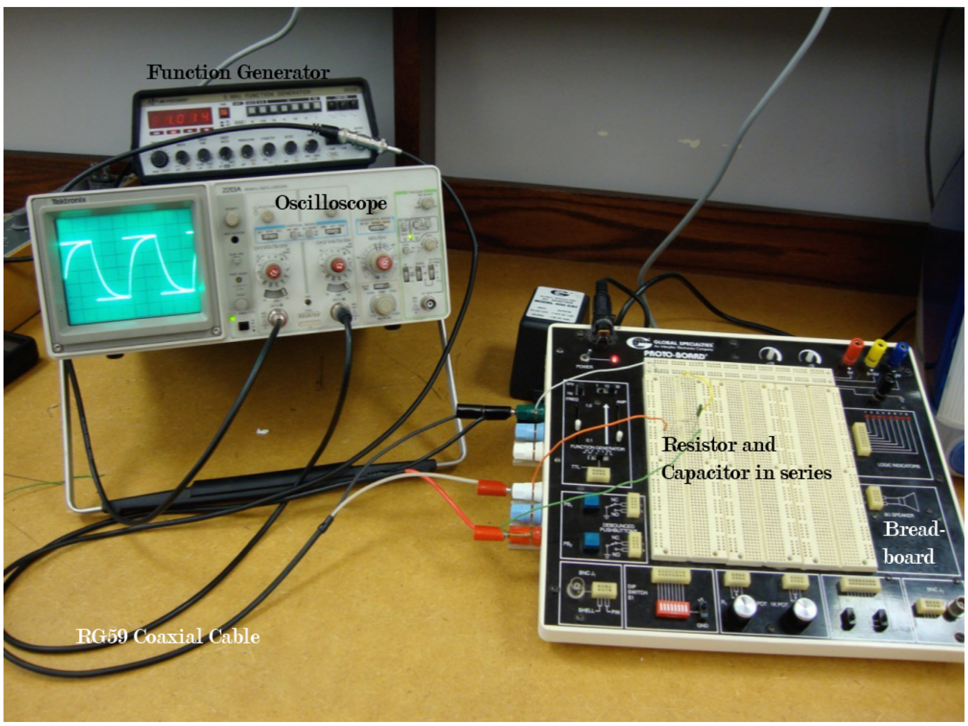

FIGURE 9. The experimental RC circuit. The resistor and capacitor are connected in series, with the power supply set to output a square wave and connected in parallel with the resistor and capacitor.

[3]A function generator can generate a square wave, a sine wave (equivalent to AC current), and a triangular wave.

In this setup (Figure 9), when the switch was closed, the capacitor was allowed to charge. After the capacitor was fully charged, the voltage was switched from 4V to 0V and the capacitor was able to discharge through the resistor. This charging and discharging was plotted on the oscilloscope by measuring the voltage across the resistor.[4] In Section 3.3, we were able to show that the voltage across the resistor should have an exponential relationship with time (Equation 3.13).

In order to verify that the discharging voltage vs. time plot was an exponential decay, the decay constant was measured and compared to the RC value. The decay constant was measured by analyzing the exponential decay curve displayed on the oscilloscope. The decay constant is the time it takes for the voltage to drop 37%. The decay constant is defined as the time constant $1/e$ seconds. This corresonds to a 37% drop in voltage. The decay constant was measured to be $0.1ms$. The resistance multiplied by capacitance was $0.097ms$. These two values are consistent within a 3% error.[5] In this experiment, only one RC circuit was created and analyzed.

Therefore, we can conclude that our mathematical model is accurate to within a 3% error of the experimental data.

5. Discussion

In this paper we proved and tested empirically fundamental laws that describe how voltage behaves in a circuit over time. The behavior was described in terms of the circuit independent variables current (I), resistance (R), and capacitance (C). In the first part of the paper (Section 3.1) basic definitions were used to derive a theoretical relationship between voltage, current and resistance. This relationship is known as Ohm's law. After the relationship was derived, it was tested empirically in a laboratory setting. By using a variable power source, we were able to successfully graph voltage against the product of current and resistance and verify Ohm's law. The perfect experimental linear relationship that was attained suggests that this testing setup had no weaknesses. However to further test and verify Ohm's law perhaps more complex circuits could be constructed. For instance,

[4] An oscilloscope allows us to see the voltage drop across the resistor over time. When a $4V$ voltage drop is seen across the resistor, the capacitor is fully charged. When a voltage drop of $0V$ is seen, the capacitor is uncharged.

[5] The equation for percent error is:

$$\frac{|\text{Accepted Value} - \text{Experimental Value}|}{\text{Actual Value}} \times 100\%.$$

voltage across multiple resistors in series and in parallel could be measured. These measurements should be in keeping with the relationship described in this paper.

In the second part of the paper a more complex theoretical voltage behavior was described using Kirchoff's law. For this part, a capacitor was introduced to the circuit altering the behavior of the voltage. In this case, the behavior of voltage over time was the variable of interest. Theoretically using Kirchoff's loop law (Section 3.3) we determined an exponential decay model for the behavior of the voltage in an RC circuit. Solving the first order differential equation of charge with respect to time did this. The solution of the differential equation yielded the charge in the circuit as a function of time, which was easily converted to voltage as a function of time. This equation (3.13) is unique because the differential equation is continuous for all time. As was the done in the first part, this relationship was tested experimentally in a lab. We successfully verified the derived relationship with empirical evidence. Once again the accuracy and precision present in the recorded data suggests that this setup had no weaknesses.

Furthermore, our model only works on a macroscopic scale. We choose to model electrical circuits on a macroscopic scale because otherwise, we cannot get a continuous solution to the differential equation. This is because charge is quantized, so microscopically the charge and the time derivative of charge are not continuous functions.

6. Conclusion

This paper has modeled the behavior of voltage in different electrical circuits over time. By doing so we hope that further research can be done that apply these mathematical relationships in new and innovative ways. For instance by understanding the behavior of voltage, perhaps modern circuits can be improved in efficiency. Also in this paper we have discussed the influence of capacitors, which are extremely important in modern circuitry. We conclude finally that the results of this analysis are essential to the overall understanding of electricity and the benefits it provides to all of us.

7. Author Contribution

A. Moorjani wrote the discussion and conclusion, as well as part of Section 3.1. He also assisted in writing the abstract and problem statement. D. Straus wrote the abstract and Section 3.3. He assisted in writing Section 3.1. He also typeset this paper in LaTeX. J. Zelenty wrote Section 3.1 and the model implementation. She also assisted in writing the problem statement. She constructed the circuits and performed the experiments.

Creating the differential equations in our mathematical model from basic principles gave us a deeper understanding of the physical meaning of the equations. Testing these equations experimentally further helped our understanding.

Appendix A. Experimental Data

A.1. **Materials.** The following materials were used:

- Global Specialties PB503 Proto Board Station,
- Single Output Variable Power Supply,
- Digital LCD Multimeter Voltmeter Ammeter Ohmmeter,
- BK Precision Oscilloscope, Analog, 2190B, 100 MHz, 2 Chan,
- Amprobe FG3C UA Sweep Function Generator,
- 10Base-2 connector - 2 x BNC - F - BNC - M - Coaxial,
- 10Base-2 BNC - M - 6" Leads with Banana Plug,
- E-Z Hook Lead w/Alligator Clips,
- 24 gauge copper wire,
- Stackable Banana plugs,
- 9900 ohm resistor,
- 9.85 nF capacitor.

The materials used were obtained from the Physics Department at the University of Chicago.

A.2. **Experimental Data for Resistor-Battery Circuit.** The following table contains the data from the resistor-battery circuit.

Voltage (V)	Current (A)
0.0683	0
0.1902	0.00001
0.3028	0.00002
0.539	0.00005
1.648	0.00016
3.846	0.00038
5.67	0.00056
7.95	0.00079
9.73	0.00097
11.88	0.00118
16.44	0.00164
20.75	0.00208
25.6	0.00256
29.79	0.003
31.78	0.00319

References

[1] HowStuffWorks autopsy: Inside a hair dryer. http://express.howstuffworks.com/autopsy-hair-dryer.htm.

[2] William E. Boyce and Richard C. DiPrima. *Elementary Differential Equations and Boundary Value Problems*. Wiley, 9 edition, October 2008.

[3] David Halliday et al. *Fundamentals of physics*. Wiley, Hoboken NJ, 7th ed., extended ed. edition, 2005.

[4] Jerrold E. Marsden and Anthony Tromba. *Vector Calculus*. W. H. Freeman, fifth edition edition, August 2003.

[5] Edward Purcell. *Electricity and magnetism*. McGraw-Hill, New York ;;Singapore, 2nd ed, international student ed. edition, 1985.

[6] Roberto Rudervall. High voltage direct current (HVDC)Transmission systems technology review paper. http://www2.internetcad.com/pub/energy/technology_abb.pdf.