MATH 237: Vector Calculus

Practice EXAM I

Answer each of the following problems. When in doubt, err on the side of more explanation rather than less–the goal of your answer is to convince the reader of your knowledge.

Calculators and collaboration with anyone else are not allowed.

Caveat: This is a practice exam. It is intended to give a sense of the format and type of question that will be on the real exam. It may be slightly longer than the true exam; problems may seem more or less difficult than those on the true exam. Moreover, this exam is not vetted as carefully as the true exam. Expect typos.

(1) *(20 points)*

(a) What does it mean to say that a vector-valued function $\mathbf{r}(t)$ is continuous? (This is a request for a definition, not a gloss.)

(b) If $\mathbf{v} \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w}$, does it follow that $\mathbf{v} = \mathbf{u}$? Explain why this is true or give a counterexample to show it is false.

(c) Find the area of the parallelogram whose vertices lie at $(1, 2, 3)$, $(4, -2, 5)$, $(5, 0, 8)$, and $(1, 1, 1)$.

(d) For vectors $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{R}^3$,

(i) Give a formula for the cosine of the angle between them.

(ii) Give a different formula for the sine of the angle between them.

(2) *(20 points)* Let $\mathbf{v} = \langle 2, 3, 6\rangle, \mathbf{w} = \langle 6, 3, 2\rangle$.

(a) What is the formula for the scalar component of $\mathbf{v}$ in the direction of $\mathbf{w}$?

(b) Justify this formula with a diagram. (You may use without argument the lemma relating the dot product and the angle between vectors.)

(c) What is the formula for the vector projection of $\mathbf{v}$ in the direction of $\mathbf{w}$?

(d) What is the relationship between scalar component of $\mathbf{v}$ in the direction of $\mathbf{w}$ and the vector projection of $\mathbf{v}$ in the direction of $\mathbf{w}$?

(e) Give two unit vectors parallel to $\mathbf{v}$.

(f) Are $\mathbf{v}$ and $\mathbf{w}$ perpendicular? Why or why not?

(3) *(20 points)*

(a) Find the angle of intersection of the lines given by

$$\mathbf{r}(t) = (2 + \sqrt{3}t)\mathbf{i} + (6 - t)\mathbf{j} + 7\mathbf{k}$$

and

$$\mathbf{s}(t) = (t + 1)\mathbf{i} + 6\mathbf{j} + 7\mathbf{k}$$

(b) Find an equation for the plane that contains these two lines.

(4) *(20 points)*

(a) Does the line with parametric equations

$$x = 1 - 4t \quad y = 3 \quad z = 2t + 2$$

intersect the plane

$$x + 2y + 2z = 5?$$

Explain why or why not.

(b) If the line and the plane intersect, find the point of intersection. If not, find the distance between them. Explain, with a diagram and/or written argument, why the distance or line you give is correct.

(5) Consider the curve with smooth parameterization

$$\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$$

(a) Write, but do not evaluate, an integral that will give the length of this curve from $t = a$ to $t = b$:

(b) Integrate (do evaluate) the function $f(x, y, z) = \sqrt{1 + 4y + 9xz}$ along this curve from $t = 0$ to $t = 2$.

(6) Consider a particle moving along a spiral curve parameterized by

$$\mathbf{r}(t) = t\mathbf{i} + t\sin t\mathbf{j} + t\cos t\mathbf{k}$$

(a) Give an equation for the line tangent to this curve at the point $(\pi, 0, -\pi)$

(b) What is the acceleration vector for this particle?

(c) Does the particle move with constant speed? Why or why not?