MATH 237: Vector Calculus

PRACTICE EXAM II

Answer each of the following problems. When in doubt, err on the side of more explanation rather than less–the goal of your answer is to convince the reader of your knowledge. All work must be your own.

Caveat: This is a practice exam. It is intended to give a sense of the format and type of question that will be on the real exam. It may be slightly longer than the true exam; problems may seem more or less difficult than those on the true exam. Moreover, this exam is not vetted as carefully as the true exam. Expect typos.

(1) (a) Let $f : \mathbb{R}^n \to \mathbb{R}$. What is the definition of

$$\lim_{\mathbf{x} \to \mathbf{x_0}} f(\mathbf{x}) = L?$$

(b) Let $f(x, y, z)$ be a function of three variables. What is the definition of $\frac{\partial f}{\partial z}$?

(c) Describe the level surfaces of the function $g(x, y, z) = x^2 + y^2 - z$?

(d) Suppose that $f(x, y)$ is defined on an open set containing (a, b). What is the definition of continuity for a function $f(x, y)$ at (a, b)?

$$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$$

(2) (a) Prove or disprove:

$$\lim_{(x,y)\to(0,0)} \frac{(x-y)^2}{x^2+y^2} = L$$

for some $L \in \mathbb{R}$.

(b) Compute $\frac{\partial T}{\partial w}$ if

$$T(x,y,z,w) = xyzw\cos(zw) + 3w^2 - 2z\ln x$$

(c) Give an equation for the plane tangent to the graph of $f(x,y) = x^2 - y^2$ at $(13, 5, 144)$.

(d) Give an equation for the plane tangent to the level surface of

$$g(x,y,z) = x^2yz - ye^x + 3\sin z = 4$$

at $(0, -7, \frac{\pi}{2})$.

(3) Find the absolute maximum and minimum of $f(x, y) = 2x^2 - 4x + y^2 - 4y + 3$ on the closed region bounded by the lines $x = 0, y = 3$, and $y = x$.

(4) Find the extreme values of $xz + 2y$ on the surface of the sphere of radius 6 centered at the origin.

(5) (a) Use the definition of the directional derivative and properties of the dot product, to show that $\nabla f(\mathbf{x_0})$ points in the direction of fastest increase of f at x_0.

(b) Find the rate of change of $f(x, y, z) = e^x yz$ at $(1, \ln 3, 4)$ in the direction of greatest increase.

(c) Compute $D_{\mathbf{u}} f(1, \ln 3, 4)$ in the direction of $2\mathbf{i}+2\sqrt{3}\mathbf{j}+\mathbf{k}$. (The function f is the same as in (b) above.)

(a) Use the plane tangent to the graph of $z = \sin(x) + \sin(y)$ at the point $(\frac{\pi}{3}, \frac{\pi}{6})$ to find an approximate value for $\sin \frac{5\pi}{6} + \sin \frac{\pi}{12}$

(b) Show that the gradient of f is perpendicular to the level surface of f for any differentiable function f.

(c) Find and use the generalized second derivative test to classify the stationary points of $f(x, y) = x^2 + y^2 - x^2y^2$.