MATH 237: Vector Calculus

PRACTICE FINAL

Answer each of the following problems. When in doubt, err on the side of more explanation rather than less–the goal of your answer is to convince the reader of your knowledge. All work must be your own.

Caveat: This is a practice exam. It is intended to give a sense of the format and type of question that will be on the real exam. It may be slightly longer than the true exam; problems may seem more or less difficult than those on the true exam. Moreover, this exam is not vetted as carefully as the true exam. Expect typos.

(1) (*20 points*)

(a) Let $f(x, y, z) = e^x(y + 2)^2 z$.

(i) What is the value of f at $(\ln 3, 3, 5)$?

(ii) What is the gradient of f at $(\ln 3, 3, 5)$?

(iii) What is the directional derivative of f in the direction of unit vector $\mathbf{u} = \langle \frac{\sqrt{2}}{2}, \frac{1}{4}, \frac{1}{4} \rangle$ at $(2, -1, 3)$?

(b) For vectors $\mathbf{v}$ and $\mathbf{w}$, with angle θ between them, express $\cos\theta$ in terms of $\mathbf{v}$ and $\mathbf{w}$.

(2) (*20 points*)

(a) Evaluate $\int_0^{\sqrt{3}} \int_{\tan^{-1} x}^{\frac{\pi}{3}} \sec y \, dy \, dx$.

(b) Find an equation for the plane that is parallel to the vector $\mathbf{v} = \langle 4, 3, -1 \rangle$ and contains the origin and the point $(2, 2, 2)$.

(3) (*20 points*)

(a) Set up (but do not evaluate) a triple integral for the volume of the solid bounded above by the cone $z^2 = x^2 + y^2$, below by the plane $z = -3$ and on the sides by the unit sphere.

(b) Find the average value of $f(x, y) = x^2 + y^2$ over the portion of the unit circle that lies in the first quadrant.

(4) (*20 points*) Consider the surface $z = f(x, y) = x^3 + 3xy + y^3$.

(a) Give an equation for the plane tangent to this surface at the point P with coordinates $(1, 1, 5)$.

(b) Find the directional derivative of f in the direction of steepest increase at P.

(c) Find and classify the critical points of this surface using the second derivative test if applicable.

(5) (*20 points*)

(a) Find the length of the spiral helix $\vec{r}(t) = t\cos(t)\mathbf{i} + t\sin(t)\mathbf{j} + t^2\mathbf{k}$ for $0 \leq t \leq 4\pi$.

(b) Find the average value $f(x, y, z)$ along C where C is the curve from part (a) and

$$f(x, y, z) = x^2 + y^2 - 2z$$

(6) (*20 points*)

(a) Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is the boundary of the rectangle traversed by proceeding in straight lines from $(2,0)$ to $(5,0)$ to $(5,3)$, to $(2,3)$ and back to $(2,0)$, and $\mathbf{F} = e^x \cos y\mathbf{i} + e^x \sin y\mathbf{j}$.

(b) Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is the unit circle traversed counterclockwise and $\mathbf{F} = \sin x e^{\cos x}\mathbf{i} + y^4\mathbf{j}$.

(7) (*20 points*)

(a) Evaluate $\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, dS$, where S is the upper half of the sphere of radius 2 centered at the origin, and $\mathbf{F} = -y\mathbf{i} + x\mathbf{j} + 6y^2z\mathbf{k}$.

(b) Evaluate $\iint_S \mathbf{F} \cdot \mathbf{n}\, dS$, out of the unit sphere, where $\mathbf{F} = 2x^3\mathbf{i} + 6yz^2\mathbf{j} + 6y^2z\mathbf{k}$.

(8) (*20 points*)

(a) How can one tell if two vectors are perpendicular?

(b) Write, but do not evaluate an integral that will give the surface area of the portion of the surface $z = e^{-(x^2+y^2)}$ that lies above the unit circle.

(c) Using the function in part (b) above, give a linear approximation of the function above for $z(x, y)$ for points (x, y) near $\ln 2, \ln 3)$.

(d) Give a geometric interpretation of the linearization formula.

(e) For a region $R \subset \mathbb{R}^n$ which can be described with continuous function boundaries in all dimensions, give an expression for the size of R.

(f) For a continuous function f defined on a region $R \subset \mathbb{R}^n$ as above, give an expression for the average value of f on R.

(g) How does the directional derivative of a multivariate function relate to the derivative of a single-variable function?

(h) For a differentiable function f, what is the gradient of f? What does it represent?

(9) (*20 points*)

(a) Find the volume of the solid with straight vertical sides bounded below by the region in the first quadrant of the xy-plane between the curves $y = x^2$ and $y = x^3$, and above by the plane $x + 2y - z = 0$.

(b) What was this course about? Write one to two paragraphs that summarize the course; this summary should reflect the most important themes, ideas and results of the semester. (It should be in mathematical English, observing usual grammatical conventions. It should not be a list. The content should reflect the content of the course, rather than personal musing.)

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