MATH 231, Chapter 1

Calculus with Functions

James Madison University

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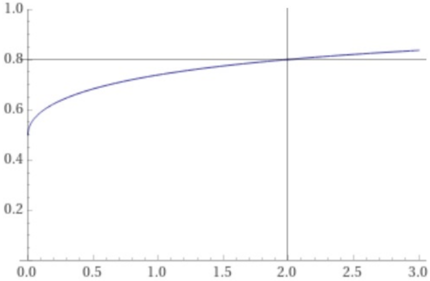

What happens to the values for $f(x)$ when x is *very close to* but not *equal to* 2?

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We say that the *limit* of f as x approaches 2 is this number, and write

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{x-2}{\sqrt{x^2+6x}-4} = ...$$

Some similar limit questions:

- $\lim_{x\to 2} \frac{x^2-4}{x-2}$
- $\lim_{x\to 2} \frac{\sqrt{x^2+4}}{x-3}$
- $\lim_{x\to 1} x^3 + x - 3$
- $\lim_{x\to 2} \frac{x^2+4}{x-2}$

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$\lim_{x\to 2} \frac{x^2-4}{x-2}$ and $\lim_{x\to 2} \frac{x-2}{\sqrt{x^2+6x}-4}$ are more interesting limits than $\lim_{x\to 2} \frac{\sqrt{x^2+4}}{x-3}$. The first two are sometimes called "indeterminate forms." For the first two limits, both the numerator and denominator are getting close to 0. " $\frac{0}{0}$ " is not a number ... it really is not anything at all other than a meaningless expression. We have to do some additional work to figure out what the limit is.

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In $\lim_{x\to 2} \frac{\sqrt{x^2+4}}{x-3}$, the numerator is getting close to $\sqrt{4+4} = 2\sqrt{2}$ and the denominator is getting close to 2-3 = -1, so the whole fraction has limit $\frac{2\sqrt{2}}{-1} = -2\sqrt{2}$.

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For now, a less formal version:

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Intuitive Description of and Notation for Limits

Suppose f is a function and L and c are real numbers.

(a) ***Limit:*** If the values of a function $f(x)$ approach L as x approaches c, then we say that L is the ***limit*** of $f(x)$ as x approaches c and we write

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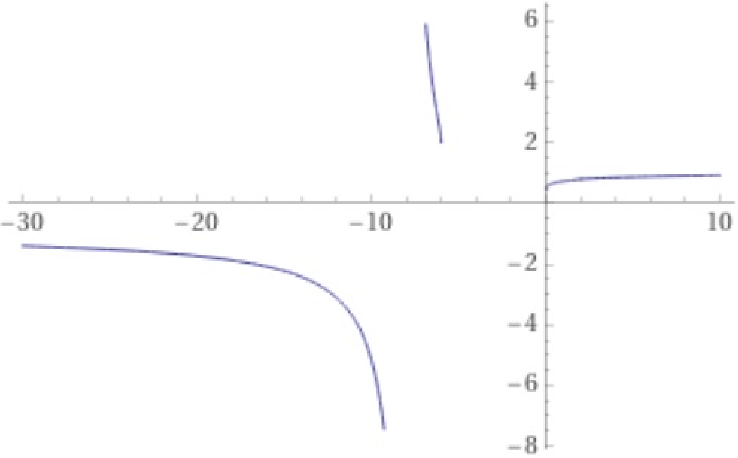
Note: The "limit" does not in any way depend on the actual value of f at the point c. $f(c)$ may not even exist, or $f(c)$ may be a number that is different from L. It is true that often $f(c) = L$ as it did in *some* of the previous examples, but this is in no way *required* in order for the limit to exist.

The graph of $f(x) = \frac{x-2}{\sqrt{x^2+6x-4}}$ on a larger scale:



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(b) ***Limit at Infinity:*** If the values of a function $f(x)$ approach L as x grows without bound, then we say that L is the ***limit*** of $f(x)$ as x approaches ∞ and we write

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(c) ***Infinite Limit:*** If the values of a function $f(x)$ grow without bound as x approaches c, then we say that ∞ is the ***limit*** of $f(x)$ as x approaches c and we write

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Horizontal Asymptotes

A nonconstant function f has a ***horizontal asymptote*** at $y = L$ if one or both of the following are true:

$$\lim_{x\to\infty} f(x) = L, \quad \text{or} \quad \lim_{x\to-\infty} f(x) = L.$$

Another example:

$$\lim_{x \to 0} \frac{\sqrt{2^{x^2} - 1}}{x}$$

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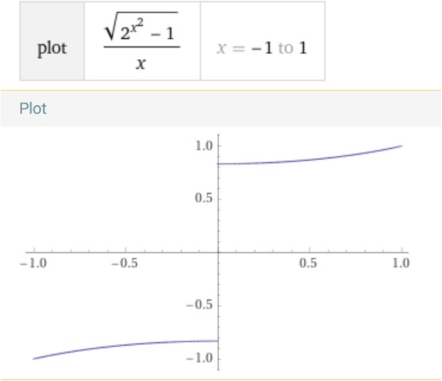

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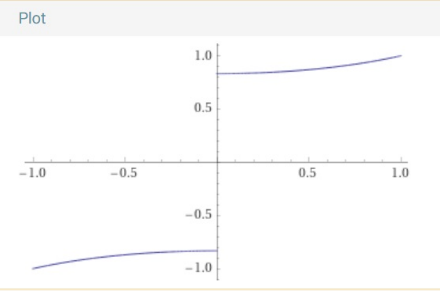
plot	$\frac{\sqrt{2^{x^2} - 1}}{x}$	$x = -1$ to 1



A similar example for which a calculator may not be needed:

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In both of the above, when we look for a limit we get different answers depending on the "direction" we are coming from.

In the case of $\frac{\sqrt{2^{x^2}-1}}{x}$, we could say that if we approach 0 using only numbers greater than 0, i.e. "from the right," then there is a limit, and we can write

$$\lim_{x \to 0^+} \frac{\sqrt{2^{x^2}-1}}{x} \approx 0.83255$$

(the actual number is $\sqrt{ln(2)}$). Similarly we have:

$$\lim_{x \to 0^-} \frac{\sqrt{2^{x^2}-1}}{x} \approx -0.83255$$

$$\lim_{x \to 2^+} \frac{|x-2|}{x-2} = 1$$

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Note again that $\lim_{x\to 0} \frac{\sqrt{2^{x^2}-1}}{x}$ and $\lim_{x\to 2} \frac{|x-2|}{x-2}$ **do not exist**.

Overall there is a limit if both the left and right limit exist **and** the left and right limit are the same number.

Intuitive Description of One-Sided Limits

If the values of a function $f(x)$ approach a value L as x approaches c from the left, we say that L is the ***left-hand limit*** of $f(x)$ as x approaches c and we write

$$\lim_{x \to c^-} f(x) = L.$$

If the values of a function $f(x)$ approach a value R as x approaches c from the right, we say that R is the ***right-hand limit*** of $f(x)$ as x approaches c and we write

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Vertical Asymptotes

A function f has a ***vertical asymptote*** at $x = c$ if one or more of the following are true:

$$\lim_{x \to c^+} f(x) = \infty, \quad \lim_{x \to c^-} f(x) = \infty, \quad \lim_{x \to c^+} f(x) = -\infty, \quad \text{or} \quad \lim_{x \to c^-} f(x) = -\infty.$$

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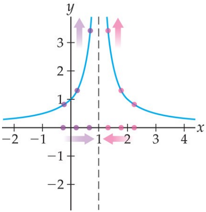
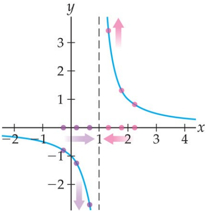
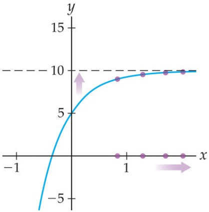
$\lim_{x\to 1} f(x) = \infty$

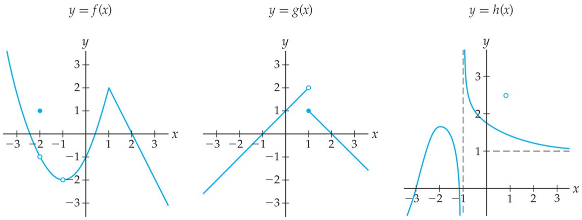


$\lim_{x\to 1^-} f(x) = -\infty, \lim_{x\to 1^+} f(x) = \infty$

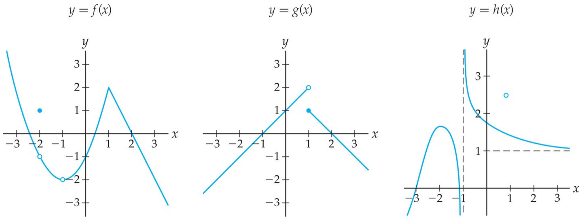


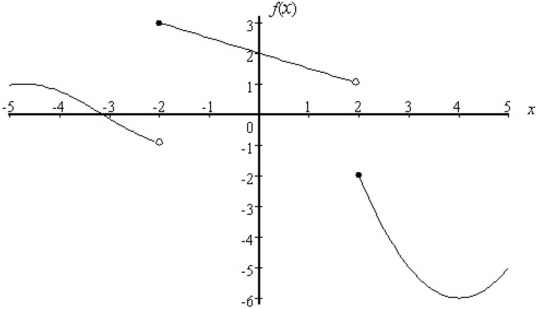
$\lim_{x\to \infty} f(x) = 10$





y = f(x)
y = g(x)
y = h(x)
y
x
−3
−2
−1
1
2
3





In the graph directly above, what if anything can you say about the limits as $x \to -2$? $x \to 0$? $x \to 2$? $x \to 2^+$? $x \to 2^-$?

Some more examples:

$$\lim_{z\to 3} \frac{z^2 - 2z - 3}{z^2 - 9} \qquad \lim_{z\to 1} \frac{z^2 - 2z - 3}{z^2 - 9}$$

$$\lim_{z\to 3} \frac{z^2 - 2z + 3}{z^2 - 9} \qquad \lim_{w\to 2} \frac{w - 2}{w^2 - 4w + 4}$$

$$\lim_{h\to 0} \frac{(3 + h)^2 + (3 + h) - 12}{h} \qquad \lim_{h\to 0} (1 + h)^{\frac{1}{h}}$$

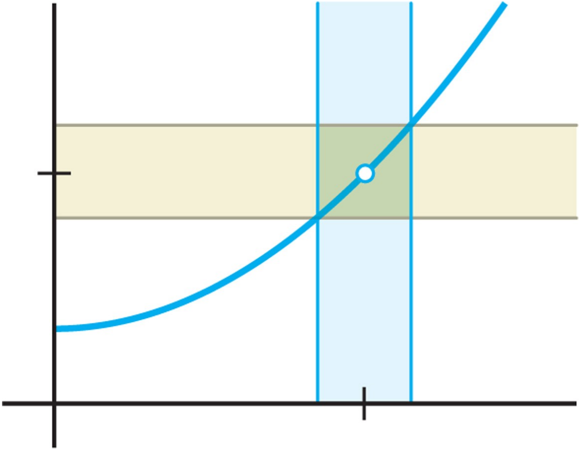
Limits as $x \to \pm\infty$ are always directly related to the existence of horizontal asymptotes. Some additional examples are below. For some of these, you should be able to reason your way to an answer without a graph. For the next to last, you may know something about the behavior of "exponentials." And the last one might take some extra algebra.

$$\lim_{z\to\infty} \frac{z^2 - 2z - 3}{z^2 - 9} \qquad \lim_{z\to\infty} \frac{3z^2 - 2z - 3}{5z^2 - 9}$$

$$\lim_{z\to\infty} \frac{z^3 - 2z - 3}{z - 9} \qquad \lim_{z\to\infty} \frac{z^2 - 2z - 3}{z^5 - 9}$$

$$\lim_{z\to\infty} \frac{z^3 - 2z - 3}{2^z} \qquad \lim_{z\to\infty} \sqrt{z^2 + z} - z$$

“Formal” definition of limits (an introduction to *error analysis*).



Formal Definition of Limit

The ***limit*** $\lim_{x \to c} f(x) = L$ means that for all $\epsilon > 0$, there exists $\delta > 0$ such that

$$\text{if } x \in (c - \delta, c) \cup (c, c + \delta), \text{ then } f(x) \in (L - \epsilon, L + \epsilon).$$

One-Sided Limits

The ***left limit*** $\lim_{x\to c^-} f(x) = L$ means that for all $\epsilon > 0$, there exists $\delta > 0$ such that

$$\text{if } x \in (c-\delta, c), \text{ then } f(x) \in (L-\epsilon, L+\epsilon).$$

The ***right limit*** $\lim_{x\to c^+} f(x) = L$ means that for all $\epsilon > 0$, there exists $\delta > 0$ such that

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For a Limit to Exist, the Left and Right Limits Must Exist and Be Equal

$\lim_{x \to c} f(x) = L$ if and only if $\lim_{x \to c^-} f(x) = L$ and $\lim_{x \to c^+} f(x) = L$.

Limits Involving Infinity

The ***infinite limit*** $\lim_{x \to c} f(x) = \infty$ means that for all $M > 0$, there exists $\delta > 0$ such that

$$\text{if } x \in (c - \delta, c) \cup (c, c + \delta), \text{ then } f(x) \in (M, \infty).$$

The ***limit at infinity*** $\lim_{x \to \infty} f(x) = L$ means that for all $\epsilon > 0$, there exists $N > 0$ such that

$$\text{if } x \in (N, \infty), \text{ then } f(x) \in (L - \epsilon, L + \epsilon).$$

The ***infinite limit at infinity*** $\lim_{x \to \infty} f(x) = \infty$ means that for all $M > 0$, there exists $N > 0$ such that

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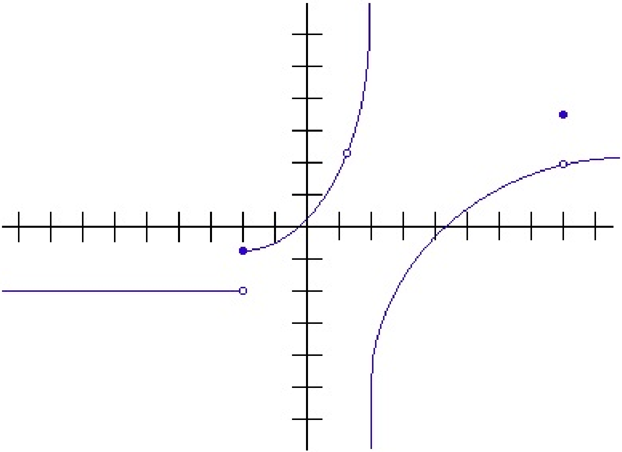
The notion of a *continuous* function or of *continuity* is fairly intuitive. Assuming that below is the graph of $y = g(x)$ for some function g, we would say that g is continuous everywhere except 4 values of x. Where is g discontinuous?



Continuity of a Function at a Point

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Left and Right Continuity at a Point

A function f is ***left continuous at*** $x = c$ if $\lim_{x \to c^-} f(x) = f(c)$ and is ***right continuous at*** $x = c$ if $\lim_{x \to c^+} f(x) = f(c)$.

Continuity of a Function on an Interval

A function f is ***continuous on an interval*** I if it is continuous at every point in the interior of I, right continuous at any closed left endpoint, and left continuous at any closed right endpoint.

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Removable, Jump, and Infinite Discontinuities

Suppose f is discontinuous at $x = c$. We say that $x = c$ is a

(a) ***removable discontinuity*** if $\lim_{x \to c} f(x)$ exists but is not equal to $f(c)$;

(b) ***jump discontinuity*** if $\lim_{x \to c^-} f(x)$ and $\lim_{x \to c^+} f(x)$ both exist but are not equal;

(c) ***infinite discontinuity*** if one or both of $\lim_{x \to c^-} f(x)$ and $\lim_{x \to c^+} f(x)$ is infinite.

The Extreme Value Theorem

If f is continuous on a closed interval $[a, b]$, then there exist values M and m in the interval $[a, b]$ such that $f(M)$ is the maximum value of $f(x)$ on $[a, b]$ and $f(m)$ is the minimum value of $f(x)$ on $[a, b]$.

The Intermediate Value Theorem

If f is continuous on a closed interval $[a, b]$, then for any K strictly between $f(a)$ and $f(b)$, there exists at least one $c \in (a, b)$ such that $f(c) = K$.

Rules for Calculating Limits of Combinations

If $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ exist, then the following rules hold for their combinations:

Constant Multiple Rule: $\lim_{x\to c} kf(x) = k \lim_{x\to c} f(x)$, for any real number k.

Sum Rule: $\lim_{x\to c}(f(x)+g(x)) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$

Difference Rule: $\lim_{x\to c}(f(x)-g(x)) = \lim_{x\to c} f(x) - \lim_{x\to c} g(x)$

Product Rule: $\lim_{x\to c}(f(x)g(x)) = (\lim_{x\to c} f(x))(\lim_{x\to c} g(x))$

Quotient Rule: $\lim_{x\to c} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$, if $\lim_{x\to c} g(x) \neq 0$

Composition Rule: $\lim_{x\to c} f(g(x)) = f(\lim_{x\to c} g(x))$, if f is continuous at $\lim_{x\to c} g(x)$

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Combinations of Continuous Functions Are Continuous

If f and g are continuous at $x = c$ and k is any constant, then the functions kf, $f+g$, $f-g$, and fg are also continuous at $x = c$.

Moreover, if $g(c) \neq 0$, then $\frac{f}{g}$ is continuous at $x = c$, and if f is also continuous at $g(c)$, then $f \circ g$ is continuous at $x = c$.

The Cancellation Theorem for Limits

If $\lim_{x \to c} g(x)$ exists, and f is a function that is equal to g for all x sufficiently close to c except possibly at c itself, then $\lim_{x \to c} f(x) = \lim_{x \to c} g(x)$.

The Cancellation Theorem for Limits

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Some more examples:

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$$\lim_{h \to 0} \frac{(3+h)^2 + (3+h) - 12}{h} \qquad \lim_{h \to 0} (1+h)^{\frac{1}{h}}$$

Limits Whose Denominators Approach Zero from the Right or the Left

(a) If $\lim_{x\to c} \frac{f(x)}{g(x)}$ is of the form $\frac{1}{0^+}$, then $\lim_{x\to c} \frac{f(x)}{g(x)} = \infty$.

(b) If $\lim_{x\to c} \frac{f(x)}{g(x)}$ is of the form $\frac{1}{0^-}$, then $\lim_{x\to c} \frac{f(x)}{g(x)} = -\infty$.

Limits Whose Denominators Become Infinite Approach Zero

(a) If $\lim_{x\to\infty} \frac{f(x)}{g(x)}$ is of the form $\frac{1}{\infty}$, then $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$.

(b) If $\lim_{x\to\infty} \frac{f(x)}{g(x)}$ is of the form $\frac{1}{-\infty}$, then $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$.

Indeterminate Forms for Limits

Each of the following is an ***indeterminate form***, meaning that a limit in one of these forms may or may not exist, depending on the situation:

$$\frac{0}{0} \qquad \frac{\infty}{\infty} \qquad 0 \cdot \infty \qquad \infty - \infty \qquad 0^0 \qquad 1^\infty \qquad \infty^0$$

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Non-Indeterminate Forms for Limits

(a) A limit in any of these forms must be equal to 0:

$$\frac{1}{\infty} \qquad \frac{0}{\infty} \qquad \frac{0}{1} \qquad 0^\infty \qquad 0^1$$

(b) A limit in any of these forms must be ∞:

$$\frac{1}{0^+} \qquad \frac{\infty}{0^+} \qquad \frac{\infty}{1} \qquad \infty+\infty \qquad \infty\cdot\infty \qquad \infty^\infty \qquad \infty^1$$

The Squeeze Theorem for Limits

If $l(x) \leq f(x) \leq u(x)$ for all x sufficiently close to c, but not necessarily at $x = c$, and if $\lim_{x \to c} l(x)$ and $\lim_{x \to c} u(x)$ are both equal to L, then $\lim_{x \to c} f(x) = L$.

Similar results hold for limits at infinity and one-sided limits.

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