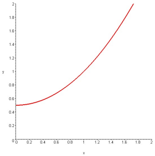
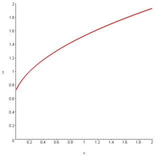
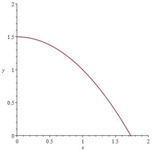
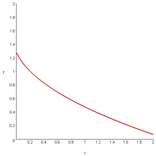
MATH 231, Chapter 3, Part 1

Calculus with Functions

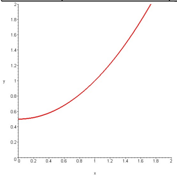
James Madison University

A function has a derivative (or is *differentiable*) at points where its graph is "smooth." In other words, the function is *continuous*, meaning that there are no breaks in the graph and no holes in the graph, and in addition there must be no sharp turns in the graph. The concept of "smooth" or "differentiable" is more or less interchangeable with the notion of "has a tangent line," with the only additional restriction being that "differentiable" requires that the tangent line is never vertical (a line with undefined slope).

The graph of such a function, if it is not a straight line, is made up of pieces that have four basic shapes, shown below. You should be able to say something about f' based on this shape, and in addition something about f'' (the derivative of the derivative). Fill in the missing pieces in the tables on the next page. An "X" indicates that one could not easily determine this unless one had more information or estimated the derivative *extremely* carefully at many points. Note that in each case, the original function is positive. The other information in the table does not depend on whether the original function is positive or negative, only on the shape of the graph.

	$+$ or $-$	$\uparrow$ or $\downarrow$	$\cup$ or $\cap$
f	$+$	$\uparrow$	$\cup$
f'			X
f''		X	X



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f	$+$		
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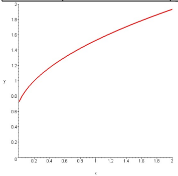

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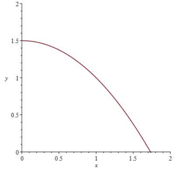

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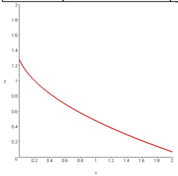

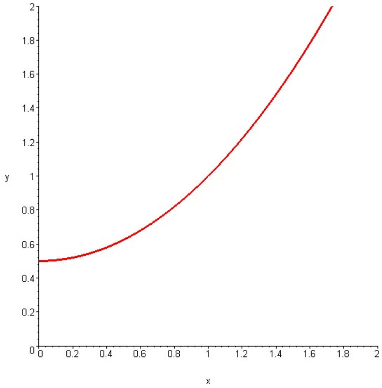
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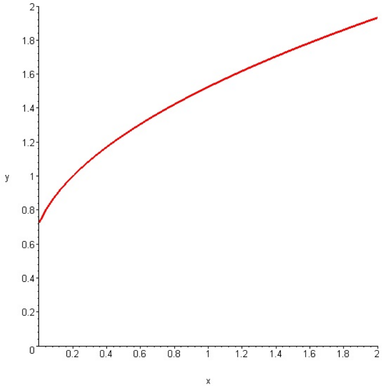
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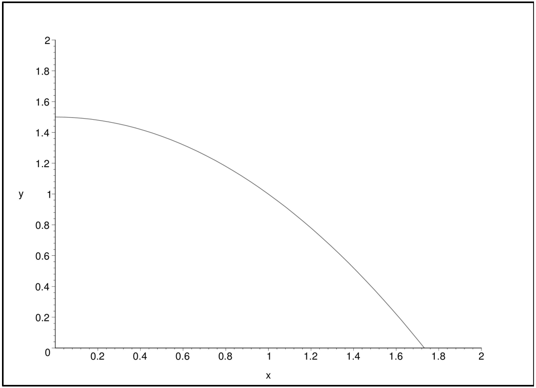

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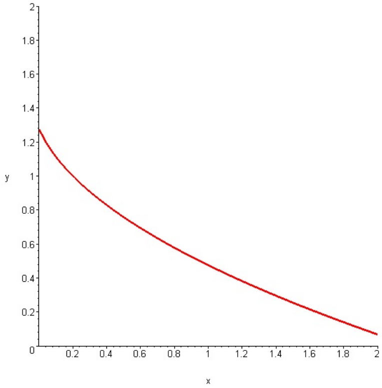
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x

Local Extrema of a Function

(a) f has a ***local maximum*** at $x = c$ if there exists some $\delta > 0$ such that $f(c) \geq f(x)$ for all $x \in (c - \delta, c + \delta)$.

(b) f has a ***local minimum*** at $x = c$ if there exists some $\delta > 0$ such that $f(c) \leq f(x)$ for all $x \in (c - \delta, c + \delta)$.

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Critical Points of a Function

A point $x = c$ in the domain of f is called a ***critical point*** of f if $f'(c) = 0$ or $f'(c)$ does not exist.

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If $x = c$ is the location of a local extremum of f, then $x = c$ is a critical point of f.

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Functions with the Same Derivative Differ by a Constant

If $f'(x) = g'(x)$ for all $x \in [a, b]$, then, for some constant C, $f(x) = g(x) + C$ for all $x \in [a, b]$.

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Rolle's Theorem

If f is continuous on $[a, b]$ and differentiable on (a, b), and if $f(a) = f(b) = 0$, then there exists at least one value $c \in (a, b)$ for which $f'(c) = 0$.

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Recall:

The Extreme Value Theorem

If f is continuous on a closed interval $[a, b]$, then there exist values M and m in the interval $[a, b]$ such that $f(M)$ is the maximum value of $f(x)$ on $[a, b]$ and $f(m)$ is the minimum value of $f(x)$ on $[a, b]$.

The Mean Value Theorem

If f is continuous on $[a, b]$ and differentiable on (a, b), then there exists at least one value $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

The Derivative Measures Where a Function is Increasing or Decreasing

Let f be a function that is differentiable on an interval I.

(a) If f' is positive in the interior of I, then f is increasing on I.

(b) If f' is negative in the interior of I, then f is decreasing on I.

(c) If f' is zero in the interior of I, then f is constant on I.

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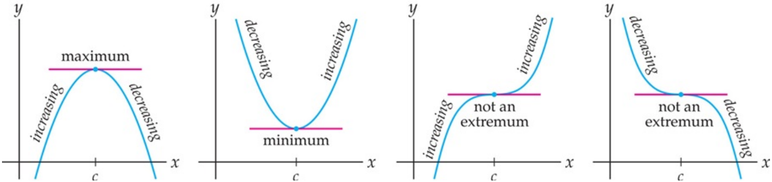
f changes from increasing to decreasing at $x = c$ | *f changes from decreasing to increasing at* $x = c$ | *f is increasing on both sides of* $x = c$ | *f is decreasing on both sides of* $x = c$



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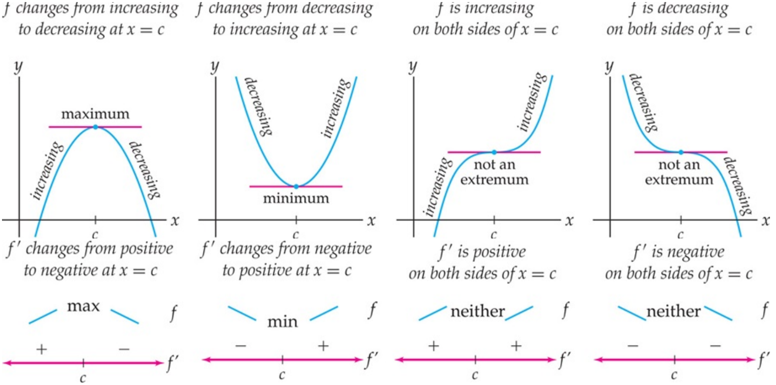
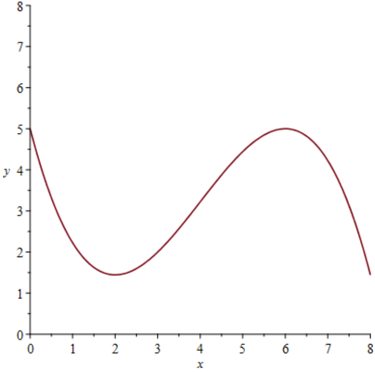


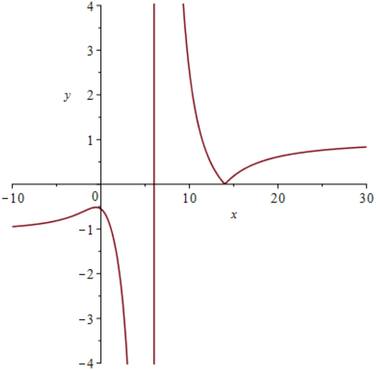
Given

$$f(x) = x^3 - 3x^2 - 9x + 27$$

Find intervals on which the graph of $y = f(x)$ is increasing? Decreasing?

Below left is the graph of f and on the right is the graph of g.





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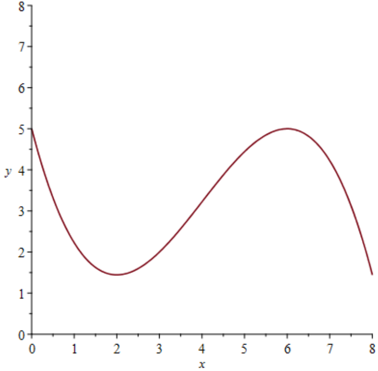

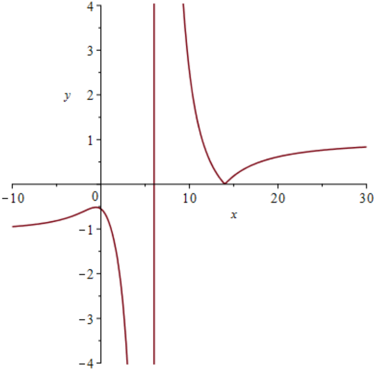

For what values of x is $f'(x) > 0$? $f'(x) < 0$? Same questionfor g?

Formally Defining Concavity

Suppose f and f' are both differentiable on an interval I.

(a) f is ***concave up*** on I if f' is increasing on I.

(b) f is ***concave down*** on I if f' is decreasing on I.

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The Second Derivative Determines Concavity

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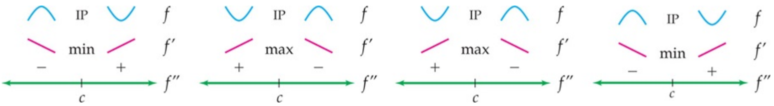

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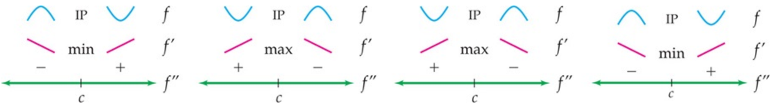

A *point of inflection* for function f is a point $(c, f(c))$ on the graph (c must be in the domain) where the "concavity" changes.

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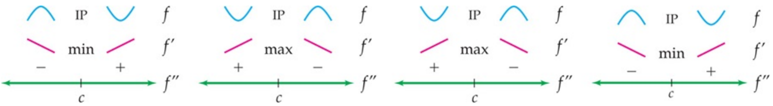

A *point of inflection* for function f is a point $(c, f(c))$ on the graph (c must be in the domain) where the "concavity" changes. We expect that this **might** occur when $f''(c)$ is ...?

Given

$$f(x) = x^3 - 3x^2 - 9x + 27$$

Find intervals on which the graph of $y = f(x)$ is increasing? Decreasing?

Find intervals where the graph of $y = f(x)$ is concave up? Concave down?

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Use this information to sketch a graph for function f.

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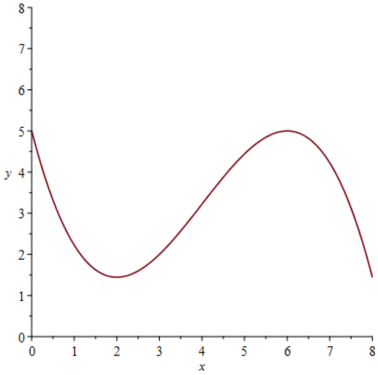

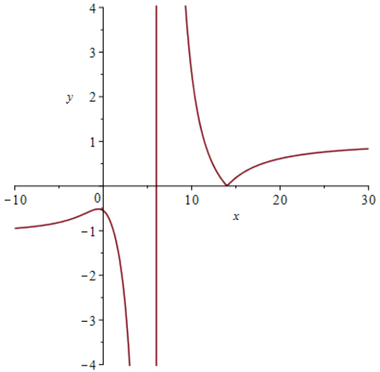

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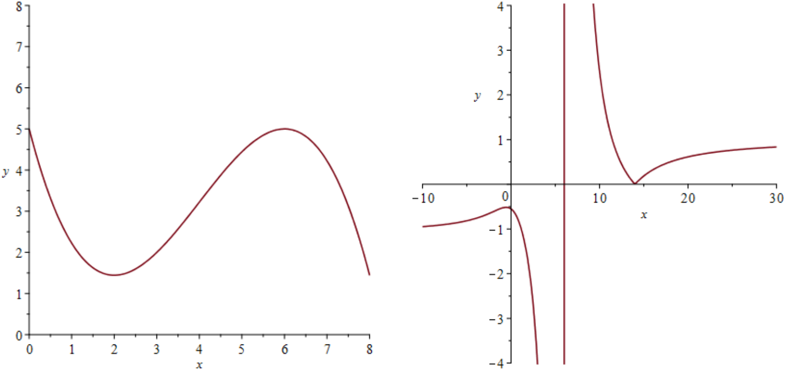

For what values of x is $f'(x) > 0$? $f'(x) < 0$? $f''(x) > 0$? $f''(x) < 0$?

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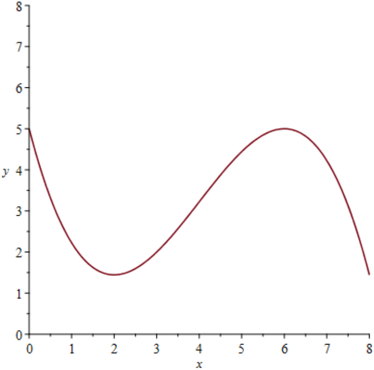

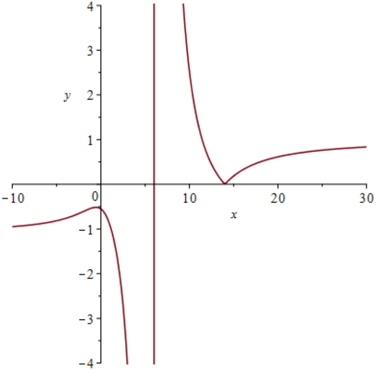

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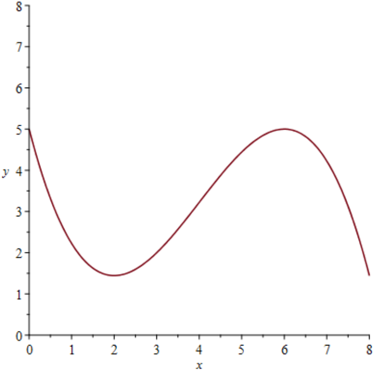

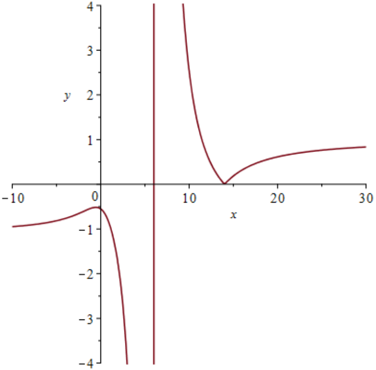

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The First-Derivative Test

Suppose $x = c$ is the location of a critical point of a function f, and let (a, b) be an open interval around c that is contained in the domain of f and does not contain any other critical points of f. If f is continuous on (a, b) and differentiable at every point of (a, b) except possibly at $x = c$, then the following statements hold.

- **(a)** If $f'(x)$ is positive for $x \in (a, c)$ and negative for $x \in (c, b)$, then f has a local maximum at $x = c$.
- **(b)** If $f'(x)$ is negative for $x \in (a, c)$ and positive for $x \in (c, b)$, then f has a local minimum at $x = c$.
- **(c)** If $f'(x)$ is positive for both $x \in (a, c)$ and $x \in (c, b)$, then f does not have a local extremum at $x = c$.
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(d) If $f'(x)$ is negative for both $x \in (a, c)$ and $x \in (c, b)$, then f does not have a local extremum at $x = c$.

The Second-Derivative Test

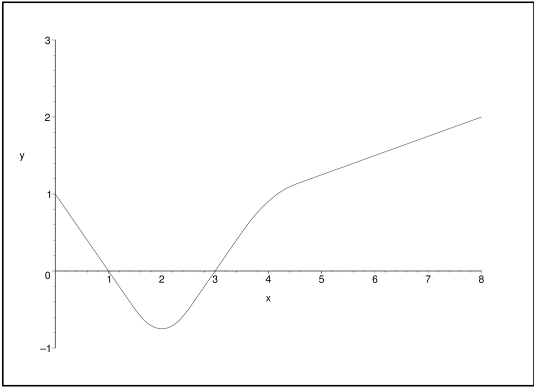
Suppose $x = c$ is the location of a critical point of a function f with $f'(c) = 0$, and suppose both f and f' are differentiable and f'' is continuous on an interval around $x = c$.

(a) If $f''(c)$ is positive, then f has a local minimum at $x = c$.

(b) If $f''(c)$ is negative, then f has a local maximum at $x = c$.

(c) If $f''(c) = 0$, then this test says nothing about whether or not f has an extremum at $x = c$.

Suppose that the graph below is $y = F'(x)$, the graph of the **derivative** of F (**not** the graph of F itself).



For which values of x is $F'(x)$ positive? Negative? Zero?
For which values of x is $F''(x)$ positive? Negative? Zero?
Given that $F(0) = 2$, sketch a rough graph of function F.

Let $f(x) = \frac{3x}{(x-1)^2}$.

Find f', f'', determine where f' and f'' are positive or negative, locate extrema and points of inflection, and compare all of this information with the graph.

At least regarding asymptotes, how would the graphs of $\frac{3x^2}{(x-1)^2}$, $\frac{3x}{(x-1)}$, $\frac{3x^2}{(x-1)}$, or $\frac{3x^2}{(x-1)^3}$ be different? (note: it is almost certainly the case that extreme values and inflection points would be quite different).