MATH 231, Chapter 3, Part 2

Calculus with Functions

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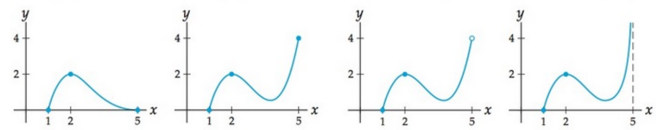
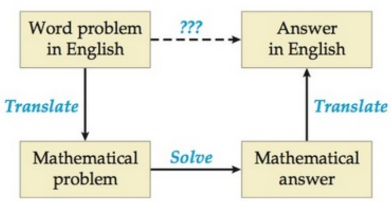
Optimization, maxima and minima: We are guaranteed that a function continuous on a finite closed interval *must* hit a maximum or minimum some where on the interval. But this can happen in various ways, including at the endpoints, and for intervals that are not closed or not finite, all manner of things are possible.

f has a global maximum on $[1,5]$ *at* $x=2$ | *g has a global maximum on* $[1,5]$ *at* $x=5$ | *h has no global maximum on* $[1,5)$ | *k has no global maximum on* $[1,5)$



So, in practice, since multiple factors must be taken into account in any problem where we are looking for a maximum or minimum value it is best to just think carefully about the situation, the domain of the function, and what exactly we are looking for.

We will be looking at several situations in which some background "story" describes a function for which we ultimately wish to find a maximum or minimum value (or possibly show that no such value exists.

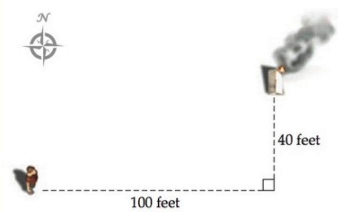


We usually begin by carefully identifying exactly what it is that we wish to maixmize or minimize. Often, that quantity depends on more than one input. If so, there may be *constraints* that allow you to find a relation between the input values and find a way to relate the quantity we wish to optimize to just one input value.

For example:

The fastest way to put out a flaming tent

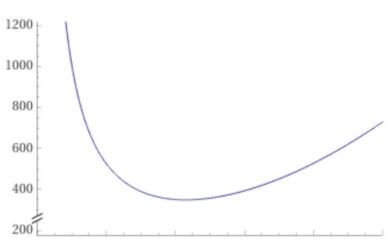
While you are on a camping trip, your tent accidentally catches fire. Luckily, you happen to be standing right at the edge of a stream with a bucket in your hand. The stream runs east–west, and the tent is 40 feet north of the stream and 100 feet farther east than you are, as shown here:



You can run only half as fast while carrying the full bucket as you can empty handed, and thus any distance travelled with the full bucket is effectively twice as long. What is the fastest way for you to get water to the tent?

Find the dimensions of a half liter can requiring the smallest amount of material.

Suggestions: Let r and h be the radius and height of the can. What is the area of the side of the can? What is the relation between r and h? What is the total area of the top and bottom? Express the surface area in terms of r alone.

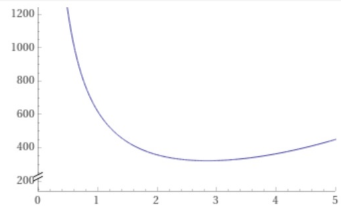


We want to design a cylindrical can which is to hold 100 cm^3. The material for the side of the can is to costs 3 cents per square cm and the material for the top and bottom will to cost 2 cents per square cm. The seams for the top and bottom of the can will cost 0.25 cents per cm while the seam up the side will cost 0.1 cents per cm. Find the dimensions of the can that will cost the least to make.

Suggestions: Let r and h be the radius and height of the can. What is the area of the side of the can? What is the relation between r and h? What is the total area of the top and bottom? What is the length of each seam?

plot $\frac{600}{r} + 4\pi r^2 + \pi r + \frac{10}{\pi r^2}$ $r = 0$ to 5

Plot



solve $\frac{\partial}{\partial r}\left(\frac{600}{r} + 4\pi r^2 + \pi r + \frac{10}{\pi r^2}\right) = 0$

Results

$r \approx -0.010610$

$r \approx 2.8419$

$r \approx -1.4782 - 2.4931\,i$

$r \approx -1.4782 + 2.4931\,i$

We will often come across siuations in which two quantities are related, we know how one of them is changing (with respect to time), and we wish to determine how the other is changing.

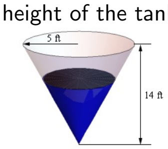
If so, we may be able to write an equation that describes the relationship between the two quantities, although in doing so the variable "time" may be hidden. That is, the equation relates two quantities that are themselves both functions of time ...'t".

To find what we wish to find we can differentiate both sides of our equation with respect to t, as in our earlier approach with so-called *implicit differentiation*.

But this is best illustrated with examples...

Air is being pumped into a spherical balloon at a rate of 5 cm^3/min. Determine the rate at which the radius of the balloon is increasing when the diameter of the balloon is 20 cm.

A tank of water in the shape of a cone is leaking water at a constant rate of 2 ft^3/hour. The radius of the tank is 5 ft and the height of the tank is 14 ft.



At what rate is the depth of the water in the tank changing when the depth of the water is 6 ft?