MATH 231, Chapter 2

Calculus with Functions

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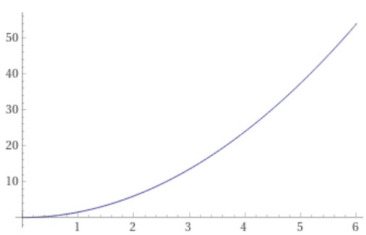
For an object in motion, we will talk about the following quantities. t represents time.

- $s(t) =$ the *position* of the object at time t, i.e. the distance from some identified point. Typical units are ft, miles, km, etc.
- $v(t) =$ the *velocity* of the object at time t, i.e. the rate at which the position is changing. Typical units are ft/sec, miles/hr, km/min, miles/sec etc.
- $a(t) =$ the *acceleration* of the object at time t, i.e. the rate at which the velocity is changing. Typical units are ft/sec per sec (ft/sec^2), miles/hr per hour (mi/hr^2), km/min per minute (km/min^2), etc.

Imagine the following: A bicycle is initially not moving, but beginning to move at time $t = 0$ (so that velocity $v(0) = 0$ but $v(t) > 0$ when $t > 0$). Also let $s(0) = 0$ (recalling that we can define the "coordinates" for our starting position anyway we want).

If the bicycle has constant acceleration, here is a *possible* formula for position s in feet after t seconds (later: how did we get this??).

$$s(t) = \frac{3}{2}t^2.$$



$$s(t) = \frac{3}{2}t^2.$$

What is the position after 3 seconds (what is $s(3)$)?

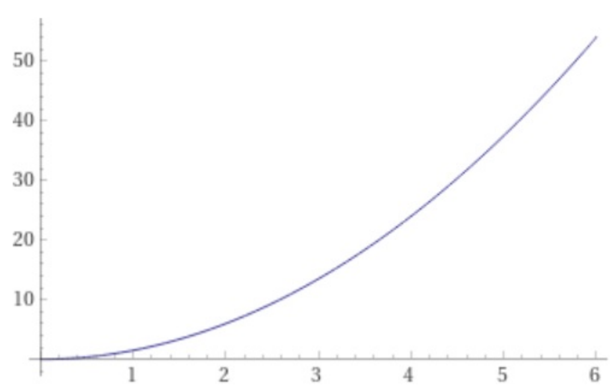
What is $s(5)$?

What is the average velocity of the bicycle between $t = 3$ and $t = 5$?

In other words, starting at $t = 3$, the average velocity over the next 2 seconds is ...?

Starting at $t = 3$ what is the average velocity ...

- ... over the next 1 second?
- ... 0.5 second?
- ... 0.1 second?
- ... 0.01 second?
- ... over the next h seconds? (Where h is an arbitrary but presumably small number.)

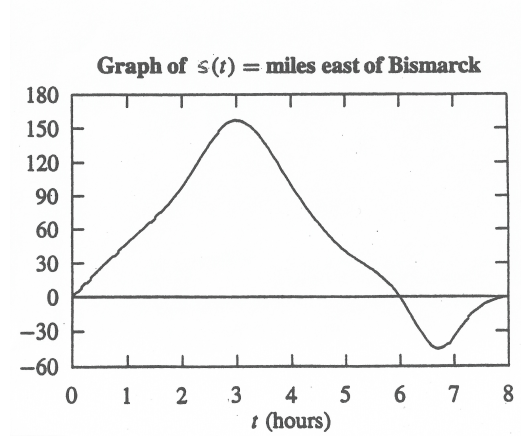


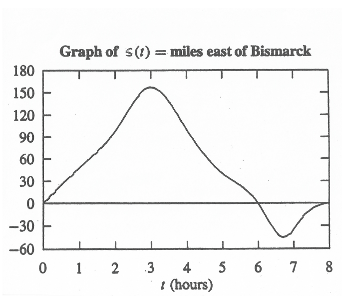
We should be able to guess the exact value for the velocity at three seconds.

Does it seem reasonable to say the following:
The average velocity between time 3 and time 3+h is ...

$$\begin{aligned}\frac{s(3+h)-s(3)}{h} &= \frac{\frac{3}{2}(3+h)^2-13.5}{h}\\ &= \frac{\frac{3}{2}(9+6h+h^2)-13.5}{h}\\ &= \frac{9h+\frac{3}{2}h^2}{h}\\ &= 9+\frac{3}{2}h\end{aligned}$$

Another example? Below is the graph of "position" for a little 8 hour automobile trek. Think about how to describe the trip. What we really want is a graph of velocity (obviously "approximate"). Bismarck ...North Dakota.





Back to our bicycle example:

$$s(t)=\frac{3}{2}t^2$$

$$\begin{aligned}\frac{s(3+h)-s(3)}{h} &= \frac{\frac{3}{2}(3+h)^2-13.5}{h}\\ &= \frac{\frac{3}{2}(9+6h+h^2)-13.5}{h}\\ &= \frac{9h+\frac{3}{2}h^2}{h}\\ &= 9+\frac{3}{2}h\end{aligned}$$

Velocity is the rate of change of position. Average velocity over an interval (of time) is the average rate of change of position.

The exact velocity (or exact rate of change of position) at time 3 is the limit of the average rate of change between time 3 and time 3+h as h gets very small. Or in symbols:

$$v(3)=\lim_{h\to 0}\frac{s(3+h)-s(3)}{h}.$$

If t is any particular time (t could be 2 or 3 or 1.67 or whatever) then the velocity at time t is

$$v(t)=\lim_{h\to 0}\frac{s(t+h)-s(t)}{h}.$$

The rate of change of s (or any function, not necessarily representing *position*) is also called the *derivative* of s and denoted s'. So we could write

$$s'(t)=\lim_{h\to 0}\frac{s(t+h)-s(t)}{h}.$$

There is nothing special about position functions. If f is *any* function, then the derivative f' (or rate of change) of f is given by

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}.$$

Note that there is no guarantee that the limit above actually exists. Sometimes limits exists, sometimes not. When and where *this* limit does exist (and thus the derivative exists) we say that the function is *differentiable*.

The Derivative of a Function at a Point

The ***derivative at*** $x = c$ of a function f is the number

$$f'(c) = \lim_{h\to 0} \frac{f(c+h) - f(c)}{h}, \quad \text{or equivalently,} \quad f'(c) = \lim_{z\to c} \frac{f(z) - f(c)}{z - c},$$

provided that this limit exists.

The Derivative of a Function

The ***derivative*** of a function f is the function f' defined by

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}, \quad \text{or equivalently,} \quad f'(x) = \lim_{z\to x} \frac{f(z) - f(x)}{z - x}.$$

The domain of f' is the set of values x for which the defining limit of f' exists.

Differentiability at a Point

A function f is ***differentiable at*** $x = c$ if $\lim_{h\to 0} \frac{f(c+h) - f(c)}{h}$ exists.

One-sided Differentiability at a Point

The ***left derivative*** and ***right derivative*** of a function f at a point $x = c$ are, respectively, equal to the following, if they exist:

$$f'_-(c) = \lim_{h\to 0^-} \frac{f(c+h) - f(c)}{h}, \qquad f'_+(c) = \lim_{h\to 0^+} \frac{f(c+h) - f(c)}{h}.$$

Differentiability Implies Continuity

If f is differentiable at $x = c$, then f is continuous at $x = c$.

Equation of the Tangent Line to a Function at a Point

The ***tangent line*** to the graph of a function f at a point $x = c$ is defined to be the line passing through $(c, f(c))$ with slope $f'(c)$, provided that the derivative $f'(c)$ exists. This line has equation

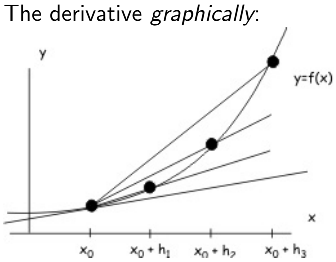
$$y = f(c) + f'(c)(x - c).$$

Local Linearity

If f has a well-defined derivative $f'(c)$ at the point $x = c$, then, for values of x near c, the function $f(x)$ can be approximated by the tangent line to f at $x = c$ with the ***linearization of f around*** $x = c$ given by

$$f(x) \approx f(c) + f'(c)(x - c).$$

The derivative *graphically*:

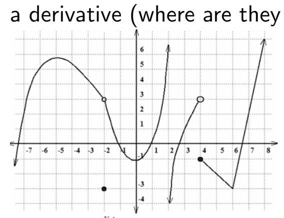


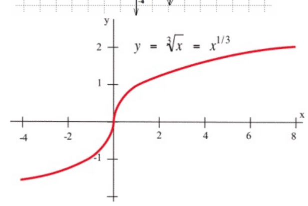
The **average** rate of change between point x_0 and point $x_0 + h$ is the slope of the line joining two points $(x_0, f(x_0))$ and $(x_0 + h, f(x_0 + h))$. This is called a *secant* line.

The actual (instantaneous) rate of change at a point is the slope of the line *tangent* to the curve at that point. The slope of the tangent line is given by f', the limit of the slopes of the secant lines:

$$f'(x_0) = \lim_{h\to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

More on this later, especially how this fits with the algebra and formulas, but we should consider what it means if a function does *not* have a derivative at a point, at least graphically. Here are a couple examples. Where do the functions in the graph fail to have a derivative (where are they not *differentiable*)?





Alternative notation(s)? If $y = f(x)$, then

$$\frac{dy}{dx} = f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

It is generally helpful to keep in mind:

$$\frac{f(x+h) - f(x)}{h} = \frac{\Delta y}{\Delta x}$$

Uh oh, lots of alternative symbols:
$f'(x), \frac{dy}{dx}, y', y'(x), \frac{df}{dx}, D_x(y), \frac{d}{dx}(f(x)), \frac{d}{dx}(y), \dot{y}, \ldots$

Other variables: If W is a function of z, for example, then derivative is $\frac{dW}{dz}$ or $W'(z)$.

Suppose that: $f(x) = x^2$? Find the derivative:

$$\begin{aligned}
f'(x) = \frac{d}{dx}(x^2) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} \\
&= \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} \\
&= \lim_{h\to 0} \frac{2xh + h^2}{h} \\
&= \lim_{h\to 0} 2x + h \\
&= 2x
\end{aligned}$$

Try another?

$$g(x) = \frac{1}{x+3}$$

Derivatives of Constant, Identity, and Linear Functions

For any real numbers k, m, and b,

(a) $\frac{d}{dx}(k) = 0$ **(b)** $\frac{d}{dx}(x) = 1$ **(c)** $\frac{d}{dx}(mx+b) = m$

We previously looked at the derivative of x^2 What is different for x^3?

$$\begin{aligned} \frac{d}{dx}(x^3) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h} \\ &= \lim_{h\to 0} \frac{x^3+3x^2h+3xh^2+h^3-x^3}{h} \\ &= \lim_{h\to 0} \frac{3x^2h+3xh^2+h^3}{h} \\ &= ?? \end{aligned}$$

What about x^4??

$$\begin{aligned} (x+h)^0 &= 1 \\ (x+h)^1 &= x+h \\ (x+h)^2 &= x^2+2xh+h^2 \\ (x+h)^3 &= x^3+3x^2h+3xh^2+h^3 \\ (x+h)^4 &= ? \end{aligned}$$

$$\begin{aligned} (x+h)^0 &= 1 \\ (x+h)^1 &= x+h \\ (x+h)^2 &= x^2+2xh+h^2 \\ (x+h)^3 &= x^3+3x^2h+3xh^2+h^3 \\ (x+h)^4 &= x^4+4x^3h+6x^2h^2+4xh^3+h^4 \\ (x+h)^5 &= x^5+5x^4h+10x^3h^2+10x^2h^3+5xh^4+h^5 \\ &\cdots \\ (x+h)^n &= ?? \end{aligned}$$

$(x+h)^n = x^n + nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \cdots + h^n$

So

$$\begin{aligned} \frac{d}{dx}(x^n) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{x^n + nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \cdots + h^n - x^n}{h} \\ &= \lim_{h\to 0} \frac{nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \cdots + h^n}{h} \\ &= \lim_{h\to 0} nx^{n-1} + \frac{n(n-1)}{2}x^{n-2}h + \cdots + h^{n-1} \\ &= ?? \end{aligned}$$

The Power Rule

For any nonzero rational number k, $\frac{d}{dx}(x^k) = kx^{k-1}$.

Derivatives of Constant Multiples and Sums of Functions

If f and g are functions and k is a constant, then for all x where the functions involved are differentiable, we have the following differentiation formulas:

Constant Multiple Rule: $(kf)'(x) = kf'(x)$

Sum Rule: $(f+g)'(x) = f'(x) + g'(x)$

Difference Rule: $(f-g)'(x) = f'(x) - g'(x)$

Find the derivative $\frac{dy}{dx}$ in each of the following:

1. $y = 3x^5 - 5x^3 - 7x + 5$
2. $y = \frac{2x^7 - 3x^3}{x^2}$
3. $y = (2x^3 + 5)^2$
4. $y = \sqrt[5]{x^7}$ (assumes more than we have explained ...)
5. $y = \frac{2x^7 - 3x^3}{x^5}$
6. $y = \sqrt[3]{5x} + x^5$
7. $y = \frac{\sqrt{x} - 3x^3}{x^2}$

Derivatives of Products and Quotients of Functions

If f and g are functions, then for all x such that both f and g are differentiable, we have the following differentiation formulas:

Product Rule: $(fg)'(x) = f'(x)\,g(x) + f(x)\,g'(x)$

Quotient Rule: $\left(\frac{f}{g}\right)'(x) = \frac{f'(x)\,g(x) - f(x)g'(x)}{(g(x))^2}$

Find f' in each of the following:

- $f(x) = (x^5 + 7x^3 - 3x + 1)(x^3 + x^2 + 2x - 1)$
- $f(x) = \frac{x^5 + x}{x^3 + 2x - 1}$
- $f(x) = \frac{x^3 + 7x + 1}{\sqrt{x} + 2}$

Unfortunately, combinations such as the following are very common.

$$h(x) = \sqrt{x^3 - 5x}$$

$$h'(x) = \ ??$$

The Chain Rule

Suppose $f(u(x))$ is a composition of functions. Then for all values of x at which u is differentiable at x and f is differentiable at $u(x)$, the derivative of f with respect to x is equal to the product of the derivative of f with respect to u and the derivative of u with respect to x.

In Leibniz notation, we write this as

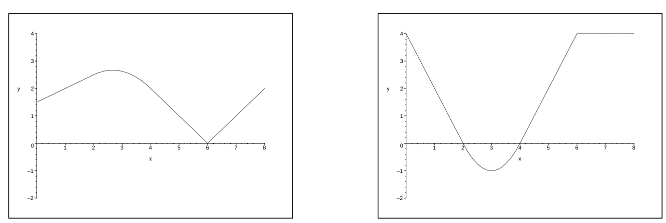
$$\frac{df}{dx} = \frac{df}{du}\,\frac{du}{dx}.$$

In "prime" notation, we write it as

$$(f \circ u)'(x) = f'(u(x))\,u'(x).$$

Find r' in each of the following:

- $r(x) = \sqrt{x^3 - 5x}$
- $r(x) = (x^4 - x^3 + 4x)^{10}$
- $r(x) = \sqrt[3]{\frac{7x^3 - 1}{2x + 1}}$
- $r(x) = (x^4 - x^3 + 4x)^{10}(x^3 + x - 1)^4$
- $r(x) = (x^3 + \sqrt{3x^2 + 2})^8$



Graphs of f (left) and g (right) are above. Let $P(x) = f(x)g(x)$, $Q(x) = f(x)/g(x)$ and $H(x) = g(f(x))$. Evaluate the following:

$f(1) =$	$g(1) =$	$f(5) =$	$g(5) =$
$f'(1) =$	$g'(1) =$	$f'(5) =$	$g'(5) =$
$P'(1) =$	$P'(5) =$	$Q'(1) =$	$H'(5) =$