MATH 231, Chapter 0

Calculus with Functions

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Intervals - describe the set of real numbers between -2 and 4, including -2 but not including 4:

$\{x \mid -2 \leq x < 4\} = [-2, 4)$
(the last, $[-2, 4)$, is *interval notation*)

Sets?
$\{5, 11, -3\}$
$\{3, 6, 9, 12, \cdots\}$

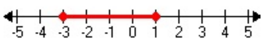
Set Notation and Elements of a Set

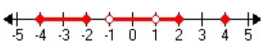
Schematically, ***set notation*** for a set S is written in the form

$$S = \{x \in \textit{category} \mid \textit{test determining whether } x \textit{ is in the set}\}.$$

An object x is an ***element*** of a set S if it is contained in S, that is, if it passes the test written in the second half of the preceding notation. When this happens, we write $x \in S$.

Write the following in interval notation:



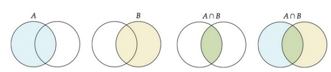


Union and Intersection

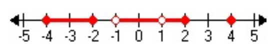
Suppose A and B are sets.

(a) The ***union*** of A and B is the set $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$.

(b) The ***intersection*** of A and B is the set $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$.



$\in$, $\subset$, $\emptyset$



Rational and Irrational Numbers
(a) A rational number is a realnumber that can be written as a quotient of the form $\frac{p}{q}$ for some integers p and q, with $q \neq 0$.

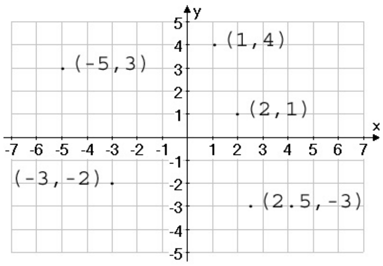
(b) An irrational number is a real number that cannot be written in the form of a rational number.

For the record, some standard sets of real numbers will be designated as follows.

- The set of natural numbers $= \{1, 2, 3, 4, \cdots\}$ - $\mathbb{N}$.
- The set of whole numbers $= \{0, 1, 2, 3, \cdots\}$ - $\mathbb{W}$.
- The set of integers $= \{0, \pm 1, \pm 2, \pm 3, \cdots\}$ - $\mathbb{Z}$
- The set of rational numbers $= \{x \mid x = \frac{p}{q}, p, q \in \mathbb{Z}, q \neq 0\}$ - $\mathbb{Q}$
- The set of real numbers - $\mathbb{R}$.

Review the basic language of sets and the meaning of symbols like $\in$, $\subset$, $\cup$, $\cap$, $\emptyset$, $^{\sim}$ and so on.

Cartesian co-ordinates



Distance between two points: $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Write the equation of a circle with center $(2, -1)$ and radius 3.

Are you familiar with the other *conic sections*?

Solving (mostly polynomial) equations.

Solve:

- $2x - 3 = 9$
- $2x^2 - 5x + 3 = 0$
- $x^2 + x = 2$
- $x^2 + x = 3$
- $\frac{2x^2-5x+3}{x^2-2} = 0$
- $\frac{2x^2-5x+3}{x^2-1} = 0$

The Quadratic Formula

If a, b, and c are real numbers, the solutions of the quadratic equation $ax^2 + bx + c = 0$ are of the form

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Formulas for Factoring Differences of Powers

For all real numbers a and b, and any positive integer n,

$$a^2 - b^2 = (a-b)(a+b)$$

$$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$

$$a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + a^{n-4}b^3 + \cdots + a^2b^{n-3} + ab^{n-2} + b^{n-1})$$

Algebraic Rules for Fractions

Let a, b, c, and d be any real numbers. Then (assuming that no denominator in any expression is zero)

$$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd} \qquad \left(\frac{a}{b}\right)\left(\frac{c}{d}\right) = \frac{ac}{bd} \qquad \frac{(a/b)}{(c/d)} = \left(\frac{a}{b}\right)\left(\frac{d}{c}\right) = \frac{ad}{bc}$$

More Algebraic Rules for Fractions

Let a, b, c, and d be any real numbers. Then (assuming that no denominator in any expression is zero)

$$\frac{a}{b} - \frac{c}{d} = \frac{ad-bc}{bd} \qquad c\left(\frac{a}{b}\right) = \frac{ac}{b} \qquad \frac{(a/b)}{c} = \frac{a}{bc} \qquad \frac{a}{(b/c)} = \frac{ac}{b}$$

Systems of equations?

$$\begin{cases} x - y = -1 \\ y + x^2 = 3 \end{cases}$$

Solving (mostly polynomial) inequalities.

Algebraic Rules for Inequalities

Suppose a, b, and c are nonzero real numbers.

(a) If $a < b$ and $c > 0$, then $ac < bc$.

(b) If $a < b$ and $c < 0$, then $ac > bc$.

(c) If $0 < a < b$, then $\frac{1}{a} > \frac{1}{b}$.

(d) If $a < b$, then $a + c < b + c$.

- $2x - 3 < 9$
- $x^2 + x \leq 2$
- $\frac{1}{x+3} \geq -2$
- $|x - 3| < 3$
- $|2x - 1| \geq 5$

Write the following in interval notation:

$\{x \; ; \; x < 5\} =$

$\{x \; ; \; x \geq -3\} =$

$\{x \; ; \; 3x - 5 \geq 0\} =$

$\{x \; ; \; 1 < -3x + 5 \leq 2\} =$

$\{x \; ; \; |2x - 7| < 3\} =$

$\{x \; ; \; |2x - 7| > 3\} =$

Functions

A ***function*** f from a set A to a set B is an assignment f that associates to each element x of the ***domain*** set A exactly one element $f(x)$ of the ***codomain***, or ***target***, set B.

Domain and Range of a Function

If f is a function between unspecified subsets of $\mathbb{R}$, then we will take the ***domain*** of f to be the largest subset of $\mathbb{R}$ for which f is defined:

$$\text{Domain}(f) = \{x \in \mathbb{R} \mid f(x) \text{ is defined}\}.$$

The ***range*** of such a function is the set of all possible outputs that it can attain:

$$\text{Range}(f) = \{y \in \mathbb{R} \mid \text{there is some } x \in \text{Domain}(f) \text{ for which } f(x) = y\}.$$

One-to-One Function

A function f is ***one-to-one*** if, for all a and b in the domain of f,

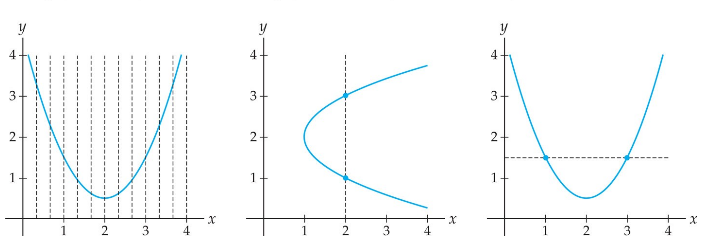
$$a \neq b \implies f(a) \neq f(b).$$

The Graph of a Function

The ***graph*** of a function f is the collection of ordered pairs $(x, f(x))$ for which x is in the domain of f. In set notation we can write

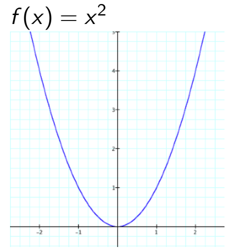
$$\text{Graph}(f) = \{(x, f(x)) \mid x \in \text{Domain}(f)\}.$$

A graph that is a function — *A graph that is not a function* — *A function, but not one-to-one*



Example - let f be the function that assigns each real number to its square.

$f : x \rightarrow x^2$
or
$f(x) = x^2$



f - "the function"
$f(x)$ the number that the function assigns to some input value x.

$f(x) = x^2$

Domain of f? Range of f?

Find

- $f(3)$
- $f(w-2)$
- $\frac{f(z)}{f(w^3+1)+2}$
- $\frac{f(x+h)-f(x)}{h}$

Suppose:

- $g(x) = \sqrt{x^2+2x}$
- $h(x) = \frac{x^2}{x^2+1}$
- $k(x) = \frac{2x}{9x^2-4}$

Domain and Range for g, h, k?

Suppose:

- $g(x) = \sqrt{x^2+2x}$

Domain and Range for g?

Suppose:

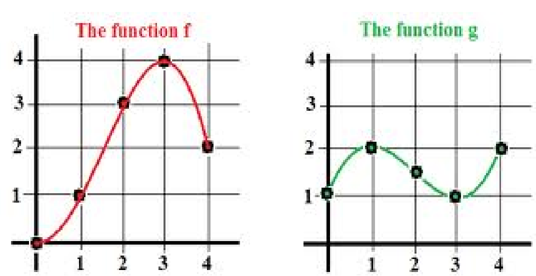
- $h(x) = \frac{x^2}{x^2+1}$

Domain and Range for h?

Suppose:

- $k(x) = \frac{2x}{9x^2-4}$

Domain and Range for k?

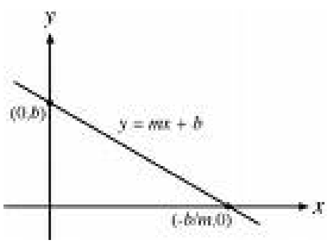


What are the domain and range of f? g?

Vocabulary	Definition	Behavior
f has a ***root*** at $x = c$	$f(c) = 0$	graph intersects the x-axis at $x = c$
f has a ***y-intercept*** at $y = b$	$f(0) = b$	graph intersects the y-axis at $y = b$
f is ***positive*** on I	$f(x) > 0$ for all $x \in I$	graph is above the x-axis on I
f is ***increasing*** on I	$f(b) > f(a)$ for all $b > a$ in I	graph moves up as we look from left to right on I
f has a ***local maximum*** at $x = c$	$f(c) \geq f(x)$ for all x near $x = c$	graph has a relative "hilltop" at $x = c$
f has a ***global maximum*** at $x = c$	$f(c) \geq f(x)$ for all $x \in$ Domain(f)	graph is the highest at $x = c$
f is ***concave up*** on I	*will state precisely in Section 3.3*	graph curves upwards on I like part of a "U"
f has an ***inflection point*** at $x = c$	*will state precisely in Section 3.3*	graph of f changes concavity at $x = c$

Algebraic functions

Type	General Form	Examples
Linear	$f(x) = mx + b$, where m and b are any real numbers	$f(x) = 2x - 1$, $f(x) = 1.4x$
Power	$f(x) = Ax^k$, where $A \neq 0$ is real and k is rational	$f(x) = 3x^3$, $f(x) = 1.7x^{-1/5}$
Polynomial	$f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$, where n is a nonnegative integer and each a_i is a real number	$f(x) = 3x^5 - 2x^3 + x - 6$
Rational	$f(x) = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials	$f(x) = \dfrac{3x^2 - 1}{x^5 + x^3 + 1}$
Other	any other algebraic function that is not one of the types listed	$f(x) = \dfrac{1 + \sqrt{x}}{1 + x}$, $f(x) = x^{2/3} + 5$



Straight lines
$y = mx + b$

$m = slope =$ "rate of change" $= \frac{vertical change}{horizontal change} = \frac{y_2 - y_1}{x_2 - x_1}$

$b = y$-axis intercept

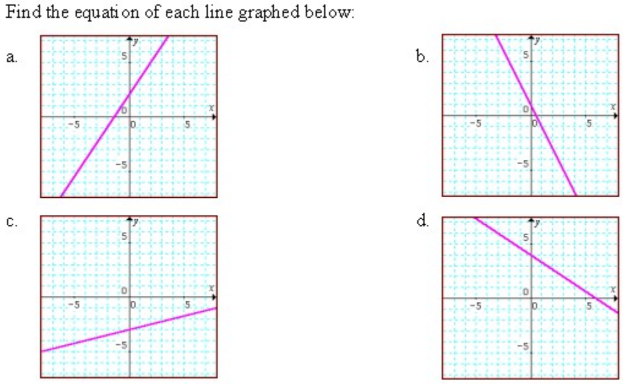
Point slope form $\quad y - y_1 = m(x - x_1)$

"General" form $\quad Ax + By + C = 0$

Horizontal line $\quad$?

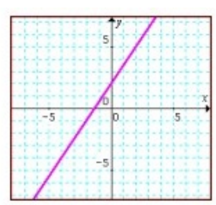
Vertical line $\quad$?

Find the equation of each line graphed below:



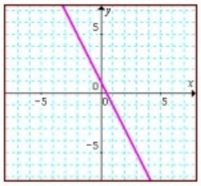
Also find:
An equation for the line parallel to a) through $(2, 1)$.
An equation for the line perpendicular to b) through $(-3, 2)$.

a.



Also find:
An equation for the line parallel to a) through $(2, 1)$.

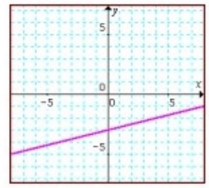
b.



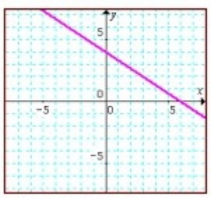
Also find:

An equation for the line perpendicular to b) through $(-3, 2)$.

c.



d.



Transcendental functions

Type	Examples
Exponential	$f(x) = 2^x$, $f(x) = 3e^{4x}$, $f(x) = 1.2(3.4)^x$
Logarithmic	$f(x) = \log_{10} x$, $f(x) = \ln x$, $f(x) = \log_2 x$
Trigonometric	$f(x) = \sin x$, $f(x) = \cos x$, $f(x) = \cot x$
Inverse Trigonometric	$f(x) = \arcsin x$, $f(x) = \cos^{-1} x$, $f(x) = \arctan x$
Other	$f(x) = x + \sin x$, $f(x) = \ln(\sqrt{x} + 12)$, $f(x) = 2^x \arctan x$

Arithmetic Combinations of Functions

Suppose f and g are functions and k is a real number.

(a) The ***constant multiple*** of f by k is the function kf defined by $(kf)(x) = kf(x)$ for all x in the domain of f.

(b) The ***sum*** of f and g is the function $f+g$ defined by $(f+g)(x) = f(x)+g(x)$ for all x in the domains of both f and g.

(c) The ***product*** of f and g is the function $f \cdot g$ defined by $(f \cdot g)(x) = f(x)g(x)$ for all x in the domains of both f and g.

(d) The ***quotient*** of f and g is the function $\frac{f}{g}$ defined by $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ for all x in the domains of both f and g with $g(x) \neq 0$.

The Composition of Two Functions

The ***composition*** of two functions f and g is the function $f \circ g$ defined by

$$(f \circ g)(x) = f(g(x))$$

for all x in the domain of g such that $g(x)$ is in the domain of f.

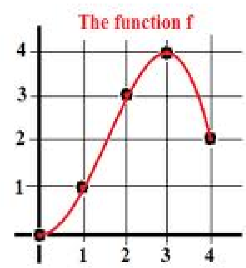
Transformation	Graphical Result	Algebraic Result
$f(x)+C$	shifts up C units if $C>0$ shifts down C units if $C<0$	$(x,y) \to (x, y+C)$
$f(x+C)$	shifts left C units if $C>0$ shifts right C units if $C<0$	$(x,y) \to (x-C, y)$
$kf(x)$	vertical stretch by k if $k>1$ vertical compression by k if $0<k<1$	$(x,y) \to (x, ky)$
$f(kx)$	horizontal compression by k if $k>1$ horizontal stretch by k if $0<k<1$	$(x,y) \to \left(\frac{1}{k}x, y\right)$
$-f(x)$	graph reflects across the x-axis	$(x,y) \to (x,-y)$
$f(-x)$	graph reflects across the y-axis	$(x,y) \to (-x,y)$

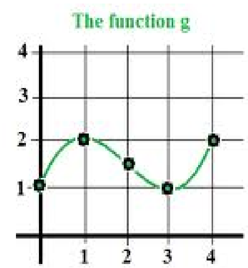
Suppose:

- $f(x) = x^2$
- $g(x) = \sqrt{x^2+2x}$
- $h(x) = \frac{x^2}{x^2+1}$
- $k(x) = \frac{2x}{9x^2-4}$

Find

- $(f+g)(x)$
- $\frac{h}{g}(x) = \frac{h(x)}{g(x)}$
- $k \circ f(x) = k(f(x))$
- $fg(x)$ (or $(f*g)(x)$)
- $g \circ f(x) = g(f(x))$
- $f \circ g(x) = f(g(x))$





Find

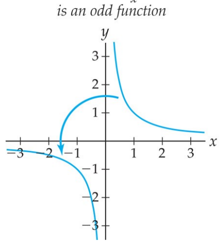
- $(f+g)(2)$
- $\frac{f}{g}(2)$
- $g \circ f(3)$
- $fg(2)$
- $f \circ g(2)$ (approx.)

Even and Odd Functions

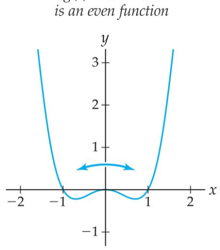
A function f is an ***even function*** if $f(-x) = f(x)$ for all x in the domain of f.

A function f is an ***odd function*** if $f(-x) = -f(x)$ for all x in the domain of f.

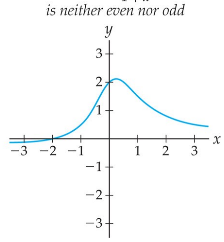
$f(x) = \frac{1}{x}$
is an odd function



$g(x) = x^4 - x^2$
is an even function



$h(x) = \frac{2+x}{1+x^2}$
is neither even nor odd



The Inverse of a Function

If f and g are functions such that

$$g(f(x)) = x, \quad \text{for all } x \text{ in the domain of } f$$
$$f(g(x)) = x, \quad \text{for all } x \text{ in the domain of } g$$

then g is the ***inverse*** of f and we denote g by f^{-1}.