MATH 510, FTC

Modern Analysis

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Some basic facts about integrals:

1. If $f_1(t)$ and $f_2(t)$ are integrable functions, then
$$\int_a^b f_1(t) + f_2(t)dt = \int_a^b f_1(t)dt + \int_a^b f_2(t)dt$$
2. If f is integrable and a, b, c are three points, then
$$\int_a^b f(t)dt = \int_a^c f(t)dt + \int_c^b f(t)dt$$
3. If f is integrable then so is $|f|$ and
$$|\int_a^b f(t)dt| \leq \int_a^b |f(t)|dt$$

Theorem If g is a continuous function and G is definied by
$$G(x) = \int_a^x g)t)dt$$
the G is differentiable and
$$G'(x) = g(x).$$

proof We need to consider
$$\lim_{\Delta x \to 0} \frac{G(x+\Delta x) - G(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\int_a^{x+\Delta x} g(t)dt - \int_a^x g(t)dt}{\Delta x}$$
and show that this limit is $g(x)$. Applying "basic fact" number 2 above with $c = x$ and $b = x + \Delta x$ we have
$$\lim_{\Delta x \to 0} \frac{G(x+\Delta x) - G(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\int_x^{x+\Delta x} g(t)dt}{\Delta x}$$
So then, we need to show that given any $\epsilon > 0$ we can find a corresponding $\delta > 0$ so that $|g(x) - \frac{\int_x^{x+\Delta x} g(t)dt}{\Delta x}| < \epsilon$ when $|\Delta x| < \delta$. But we know that since g is continuous, given $\epsilon > 0$ we can find a corresponding $\delta > 0$ so that $|g(t) - g(x)| < \epsilon$ when $|x - t| < \delta$. Thus:

$$\begin{aligned}
|g(x) - \frac{\int_x^{x+\Delta x} g(t)dt}{\Delta x}| &= |g(x) - \frac{\int_x^{x+\Delta x} g(t) - g(x) + g(x)dt}{\Delta x}| \\
&= |g(x) - \frac{\int_x^{x+\Delta x} g(t) - g(x)dt + g(x)\int_x^{x+\Delta x} dt}{\Delta x}| \\
&= |g(x) - \frac{\int_x^{x+\Delta x} g(t) - g(x)dt + g(x)\cdot \Delta x}{\Delta x}| \\
&= |\frac{\int_x^{x+\Delta x} g(t) - g(x)dt}{\Delta x}| \\
&\leq \frac{\int_x^{x+\Delta x} |g(t) - g(x)|dt}{\Delta x} < \frac{\int_x^{x+\Delta x} \epsilon \; dt}{\Delta x} = \epsilon
\end{aligned}$$

Since g is continuous, given $\epsilon > 0$ we can find a corresponding $\delta > 0$ so that $|g(t) - g(x)| < \epsilon$ when $|x - t| < \delta$.