MATH 510, Notes 1

Modern Analysis

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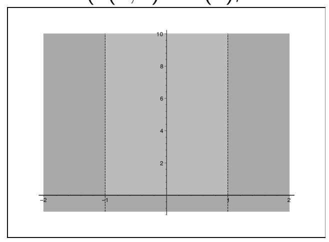
Fourier's problem – the lowdown. For the region in the (infinite) strip below with x between -1 and 1 and $w > 0$, Fourier (as an illustration of how to find a solution) wanted to compute the temperature $z(x, w)$ for any point in the region, assuming that the temperature was held at 0 on the two vertical sides ($z(\pm 1, w) = 0$) and there was some known temperature distribution $f(x)$ on the bottom ($z(x, 0) = f(x)$, i.e. when $w = 0$).



From physics: temperature distribution z in a steady state satisfies Laplace's equation

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial w^2} = 0$$

Through some typical 19th-century style manipulation (!), Fourier noted that any function of the form

$$e^{\frac{-(2k-1)\pi w}{2}} \cos(\frac{(2k-1)\pi x}{2})$$

is a solution to Laplace's equation,and in fact any sum of these functions (with constants possibly multiplying each one)is also solution. That is:

$$z(x, w) = a_1 e^{\frac{-\pi w}{2}} \cos(\frac{\pi x}{2}) + a_2 e^{\frac{-3\pi w}{2}} \cos(\frac{3\pi x}{2}) + \cdots + a_n e^{\frac{-(2n-1)\pi w}{2}} \cos(\frac{(2n-1)\pi x}{2})$$

The values along the x axis when $w = 0$, $z(x, 0)$ are then given by

$$z(x, 0) = a_1 \cos(\frac{\pi x}{2}) + a_2 \cos(\frac{3\pi x}{2}) + \cdots + a_n \cos(\frac{(2n-1)\pi x}{2})$$

Given any heat distribution function $f(x)$ for the x axis, the problem would then be solved if we can find numbers $a_1, a_2, \cdots a_n$ so that the above formula gives us the appropriate $f(x)$.

$$z(x, w) = a_1 e^{\frac{-\pi w}{2}} \cos(\frac{\pi x}{2}) + a_2 e^{\frac{-3\pi w}{2}} \cos(\frac{3\pi x}{2}) + \cdots + a_n e^{\frac{-(2n-1)\pi w}{2}} \cos(\frac{(2n-1)\pi x}{2})$$

$$f(x) = z(x, 0) = a_1 \cos(\frac{\pi x}{2}) + a_2 \cos(\frac{3\pi x}{2}) + \cdots + a_n \cos(\frac{(2n-1)\pi x}{2})$$

But the above is *not* possible even with something as simple as $f(x) = 1$. (Why?) Not deterred, Fourier suggested infinite sums instead, and went on to provide an argument for why an infinite sum *could* work if

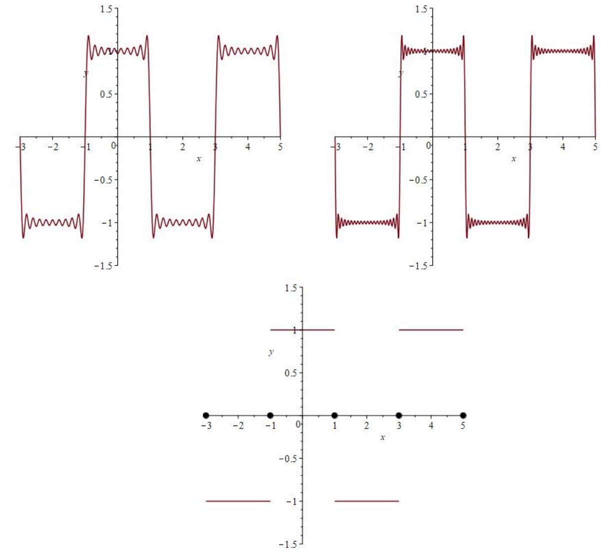
$$a_n = \frac{4}{\pi}\frac{(-1)^{n-1}}{2n-1}$$

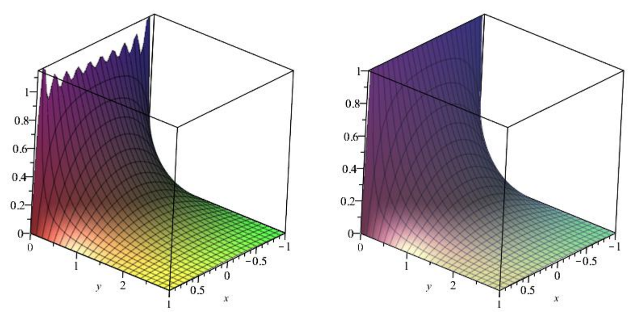
so that we might have

$$1 = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}\cos(\frac{(2n-1)\pi x}{2})$$

and

$$z(x,w) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}e^{\frac{-(2n-1)\pi w}{2}}\cos(\frac{(2n-1)\pi x}{2})$$





It is somewhat tempting to get into Fourier's original argument and not bad as a review of some calculus and algebra, not to mention an introduction to the sort of thinking that was typical in 19th century mathematics research but seems a little odd now. But going through this sort of thing is somewhat deadly in an online discussion, and being able to reproduce Fourier's argument is not essential to what comes next. The author provides a bit more detail on what is in the text at the following... also accessible from our website at the Textbook Supplementary Materials link:

http://www.macalester.edu/aratra/edition2/chapter1/chapt1a.pdf

and a bit more on Laplace's equation and why it relates to heat transfer:
http://www.macalester.edu/aratra/edition2/chapter1/chapt1b.pdf
although you can pick that up in physics or differential equations books, if you like.

The important thing for us is that these sorts of problems raised all kinds of questions about why, how, and whether this sort of thinking would lead to useful solutions. It was not at all obvious when it would all work out OK and when it might lead to some bogus solution.

So ... that is our primary lesson from Chapter 1: The proposed solutions to practical problems raised questions that could only be answered by delving much deeper into the theoretical foundations of what was being done. Did this really make sense? When does it make sense and when will it go astray?

We also want to get a bit familiar with Mathematica and Wolfram Alpha.