# MATH 510, Notes 3

Modern Analysis

James Madison University

## The Nested Interval Principle

Given an increasing sequence $\{x_1 \leq x_2 \leq x_3 \leq \cdots\}$ and a decreasing sequence $\{y_1 \geq y_2 \geq y_3 \geq \cdots\}$ such that every $y_j$ is larger than every $x_k$, but the difference $y_n - x_n$ can be made arbitrarily small by taking $n$ sufficiently large, there is **exactly one** real number that is greater than or equal to every $x_n$ and less than or equal to every $y_n$.

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You were not assigned Problem 2.2.9. Nonetheless, that problem would demonstrate that for $|x| < 1$

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This is equivalent to

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = \ln(1+x)$$

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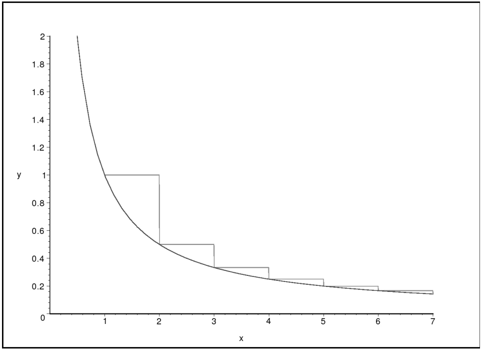
$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

does **not** approach a target value.

If you can remember at least one way to show this, it probably involves the comparison

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} > \int_1^{n+1} \frac{1}{x} dx$$



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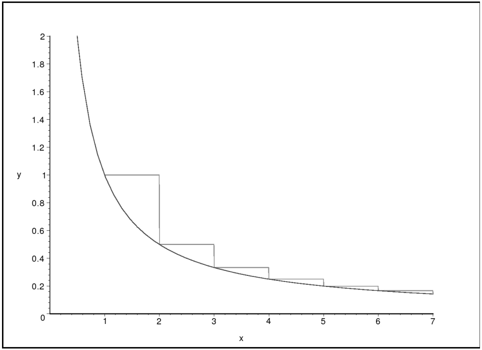


(...related to the so-called integral test)

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We know something about $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ and $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ as well as geometric series such as $1 + \frac{2}{3} + (\frac{2}{3})^2 + (\frac{2}{3})^3 + (\frac{2}{3})^4 + \cdots$.

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But other than small variations on the above that is about it.

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You might know the sum of the following series (or maybe not):
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When we get to it, we will not have too much trouble showing that this series has a target value, but determining that target value is a much more serious endeavor.

The series above are not nearly as complicated as

$$\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}\cos(\frac{(2n-1)\pi x}{2}) = \frac{4}{\pi}(\cos(\frac{\pi x}{2}) - \frac{1}{3}\cos(\frac{3\pi x}{2}) + \cdots)$$

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A type of series that was already known and studied in the time of Fourier (and they **are** simpler to deal with):

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

Taylor's formula: Suppose that the domain of $f$ is an open interval, that $a$ is a number in the domain, and that $f$ has derivatives (at least through the $n-$th). If $x$ is another number in the domain then there is a number $c$ between $a$ and $x$ so that:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n-1)}(a)}{(n-1)!}(x-a)^{n-1} + \frac{f^{(n)}(c)}{n!}(x-a)^n$$

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The piece at the end $D_n(a,x) = \frac{f^{(n)}(c)}{n!}(x-a)^n$ is called the (LaGrange) **remainder**.

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For this to work the series must first of all have a target value, but to be sure that the target value really is $f(x)$ and not some other function we need something more: ??

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