MATH 510, Notes 3

Modern Analysis

James Madison University

The Nested Interval Principle

Given an increasing sequence $\{x_1 \le x_2 \le x_3 \le \cdots\}$ and a decreasing sequence $\{y_1 \ge y_2 \ge y_3 \ge \cdots\}$ such that every y_j is larger than every x_k, but the difference $y_n - x_n$ can be made arbitrarily small by taking n sufficiently large, there is **exactly one** real number that is greater than or equal to every x_n and less than or equal to every y_n.

The set $\mathbb{R}$ of real numbers has the nested interval property. What about the set of rational numbers?

We know that

$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$$

for $|x| < 1$ (using the fact that
$1 + x + x^2 + .. + x^{n-1} = \frac{1}{1-x} - \frac{x^n}{1-x}$, etc...)

You were not assigned Problem 2.2.9. Nonetheless, that problem would demonstrate that for $|x| < 1$

$$x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots = -\ln(1-x)$$

This is equivalent to

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = \ln(1+x)$$

A few other facts:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

converges to a target value, although curiously the "order" in which we do the sum affects the answer.

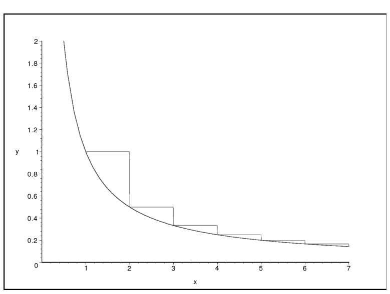
We think that the target value is $\log(2)$. The previous slide seems to imply this even if it does not prove it (why? and why not?). We can also show that

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

does **not** approach a target value.

If you can remember at least one way to show this, it probably involves the comparison

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} > \int_1^{n+1} \frac{1}{x} dx$$



(...related to the so-called integral test)

It is good to know at least one other way to show this, not depending on integrals.Think about partial sums where the number of terms is a power of 2:

$$1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \cdots + \frac{1}{8}\right) + \left(\frac{1}{9} + \cdots + \frac{1}{16}\right) + \left(\frac{1}{17} + \cdots + \frac{1}{32}\right)$$

We will later get back to the question of when a given series of numbers has a target value. As far as things that we really *know*, we have a fairly limited repertoire so far:

We know something about $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ and $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ as well as geometric series such as $1 + \frac{2}{3} + (\frac{2}{3})^2 + (\frac{2}{3})^3 + (\frac{2}{3})^4 + \cdots$.

But other than small variations on the above that is about it.

Also, for the record, the phrase “the series has a target value” is of course equivalent to “the series converges” or “the series has a limit.”

Often it is much easier to show that a series has a target value than to actually determine *what is* the target value. For geometric series the two went hand in hand. But recall for the alternating harmonic series we were only *guessing* that the target value is log(2). (“log” is the same as natural log in this course, by the way).

You might know the sum of the following series (or maybe not): $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots = \frac{\pi^2}{6}$
When we get to it, we will not have too much trouble showing that this series has a target value, but determining that target value is a much more serious endeavor.

The series above are not nearly as complicated as

$$\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}\cos(\frac{(2n-1)\pi x}{2}) = \frac{4}{\pi}(\cos(\frac{\pi x}{2}) - \frac{1}{3}\cos(\frac{3\pi x}{2}) + \cdots)$$

when we are trying to determine if there is a target value (convergence) or determining what is the target value.

A type of series that was already known and studied in the time of Fourier (and they **are** simpler to deal with):

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

Taylor's formula: Suppose that the domain of f is an open interval, that a is a number in the domain, and that f has derivatives (at least through the n–th). If x is another number in the domain then there is a number c between a and x so that:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 +$$

$$\cdots + \frac{f^{(n-1)}(a)}{(n-1)!}(x-a)^{n-1} + \frac{f^{(n)}(c)}{n!}(x-a)^n$$

The piece at the end $D_n(a,x) = \frac{f^{(n)}(c)}{n!}(x-a)^n$ is called the (LaGrange) **remainder**.

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 +$$

$$\cdots + \frac{f^{(n-1)}(a)}{(n-1)!}(x-a)^{n-1} + \frac{f^{(n)}(c)}{n!}(x-a)^n$$

What is hoped, of course, is that

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

For this to work the series must first of all have a target value, but to be sure that the target value really is $f(x)$ and not some other function we need something more: ??

It turns out that the following Taylor series *do* converge to the appropriate function at least for some values of x:
$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$

$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$

$\tan(x) = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \cdots$

Using $(x-1)$ instead of x:
$\ln(x) = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots$

$\ln(x+1) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$