# MATH 510, Notes 4

Modern Analysis

James Madison University

$$\lim_{x \to a} f(x) = T$$

means that for any numbers $M$ and $L$ with $M > T$ and $L < T$ then we can force $f(x)$ to be between $L$ and $M$ by requiring that $x$ is sufficiently close to (but not equal to) $a$.

$$\lim_{x \to a} f(x) = T$$

means that for any numbers $M$ and $L$ with $M > T$ and $L < T$ then we can force $f(x)$ to be between $L$ and $M$ by requiring that $x$ is sufficiently close to (but not equal to) $a$.

Note: "but not equal to a" recognizes that we just *do not care* what happens at point $a$ itself.

$$\lim_{x \to a} f(x) = T$$

means that for any numbers $M$ and $L$ with $M > T$ and $L < T$ then we can force $f(x)$ to be between $L$ and $M$ by requiring that $x$ is sufficiently close to (but not equal to) $a$.

Note: "but not equal to a" recognizes that we just *do not care* what happens at point $a$ itself.

For many of the examples you saw in calculus $\lim_{x \to a} f(x)$ was equal to $f(a)$. But you also saw more interesting examples such as $\lim_{x \to 0} \frac{sin(x)}{x}$ or $\lim_{x \to 0} \frac{1 - cos(x)}{x^2}$ where the formula in question is not even defined at $x = a$.

$$\lim_{x \to a} f(x) = T$$

means that for any numbers $M$ and $L$ with $M > T$ and $L < T$ then we can force $f(x)$ to be between $L$ and $M$ by requiring that $x$ is sufficiently close to (but not equal to) $a$.

Note: "but not equal to a" recognizes that we just *do not care* what happens at point $a$ itself.

For many of the examples you saw in calculus $\lim_{x \to a} f(x)$ was equal to $f(a)$. But you also saw more interesting examples such as $\lim_{x \to 0} \frac{sin(x)}{x}$ or $\lim_{x \to 0} \frac{1 - cos(x)}{x^2}$ where the formula in question is not even defined at $x = a$.

As applied to defining the derivative, the limits in question are all of the more "interesting" variety.

The **derivative** of function $f$ at point $a$ exists and has the value $f'(a)$ if for any numbers $M$ and $L$ with $M > f'(a)$ and $L < f'(a)$ we can force $\frac{f(x)-f(a)}{x-a}$ to be between $L$ and $M$ by requiring that $x$ is sufficiently close to but not equal to $a$.

The **derivative** of function $f$ at point $a$ exists and has the value $f'(a)$ if for any numbers $M$ and $L$ with $M > f'(a)$ and $L < f'(a)$ we can force $\frac{f(x)-f(a)}{x-a}$ to be between $L$ and $M$ by requiring that $x$ is sufficiently close to but not equal to $a$.

That is, $f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$

The **derivative** of function $f$ at point $a$ exists and has the value $f'(a)$ if for any numbers $M$ and $L$ with $M > f'(a)$ and $L < f'(a)$ we can force $\frac{f(x)-f(a)}{x-a}$ to be between $L$ and $M$ by requiring that $x$ is sufficiently close to but not equal to $a$.

That is, $f'(a) = \lim_{x \to a} \frac{f(x)-f(a)}{x-a}$

"Differentiable" = "has a derivative."

Or...(Cauchy):

Or...(Cauchy):

Let $E(x,a) = f'(a) - \frac{f(x)-f(a)}{x-a}$

The **derivative** of function $f$ at point $a$ exists and has the value $f'(a)$ if for any number $\epsilon > 0$ it is possible to find a corresponding $\delta > 0$ so that if $0 < |x-a| < \delta$ then this forces

$$|E(x,a)| < \epsilon.$$

One of many theorems about derivatives:

### Theorem

*Suppose that two functions $f$ and $g$ have derivatives at point $a$ and that $F = f + g$. Then $F$ also has a derivative at $a$ and $F'(a) = f'(a) + g'(a)$.*

One of many theorems about derivatives:

### Theorem

*Suppose that two functions $f$ and $g$ have derivatives at point $a$ and that $F = f + g$. Then $F$ also has a derivative at $a$ and $F'(a) = f'(a) + g'(a)$.*

What we need:

One of many theorems about derivatives:

### Theorem

*Suppose that two functions $f$ and $g$ have derivatives at point $a$ and that $F = f + g$. Then $F$ also has a derivative at $a$ and $F'(a) = f'(a) + g'(a)$.*

What we need:Let

$$E(x, a) = \frac{F(x) - F(a)}{x - a} - (f'(a) + g'(a))$$

Given any positive number $\epsilon$, we must explain how to find a number $\delta$ so that $|E(x, a)| < \epsilon$ when $0 < |x - a| < \delta$.

$$|E(x,a)| \quad = \quad |\frac{F(x)-F(a)}{x-a} - (f'(a)+g'(a))|$$

$$
\begin{aligned}
|E(x,a)| &= \left|\frac{F(x)-F(a)}{x-a} - (f'(a)+g'(a))\right| \\
&= \left|\frac{f(x)+g(x)-f(a)-g(a)}{x-a} - (f'(a)+g'(a))\right|
\end{aligned}
$$

$$
\begin{aligned}
|E(x,a)| &= \left|\frac{F(x)-F(a)}{x-a} - (f'(a)+g'(a))\right| \\
&= \left|\frac{f(x)+g(x)-f(a)-g(a)}{x-a} - (f'(a)+g'(a))\right| \\
&= \left|\frac{f(x)-f(a)}{x-a} - f'(a) + \frac{g(x)-g(a)}{x-a} - g'(a)\right|
\end{aligned}
$$

$$
\begin{aligned}
|E(x,a)| &= \left|\frac{F(x)-F(a)}{x-a} - (f'(a)+g'(a))\right| \\
&= \left|\frac{f(x)+g(x)-f(a)-g(a)}{x-a} - (f'(a)+g'(a))\right| \\
&= \left|\frac{f(x)-f(a)}{x-a} - f'(a) + \frac{g(x)-g(a)}{x-a} - g'(a)\right| \\
&\leq \left|\frac{f(x)-f(a)}{x-a} - f'(a)\right| + \left|\frac{g(x)-g(a)}{x-a} - g'(a)\right|
\end{aligned}
$$

$$
\begin{aligned}
|E(x,a)| &= \left|\frac{F(x)-F(a)}{x-a} - (f'(a)+g'(a))\right| \\
&= \left|\frac{f(x)+g(x)-f(a)-g(a)}{x-a} - (f'(a)+g'(a))\right| \\
&= \left|\frac{f(x)-f(a)}{x-a} - f'(a) + \frac{g(x)-g(a)}{x-a} - g'(a)\right| \\
&\leq \left|\frac{f(x)-f(a)}{x-a} - f'(a)\right| + \left|\frac{g(x)-g(a)}{x-a} - g'(a)\right|
\end{aligned}
$$

Both $f$ and $g$ meet the requirements of Cauchy's definition.

$$
\begin{aligned}
|E(x,a)| &= \left|\frac{F(x)-F(a)}{x-a} - (f'(a)+g'(a))\right| \\
&= \left|\frac{f(x)+g(x)-f(a)-g(a)}{x-a} - (f'(a)+g'(a))\right| \\
&= \left|\frac{f(x)-f(a)}{x-a} - f'(a) + \frac{g(x)-g(a)}{x-a} - g'(a)\right| \\
&\leq \left|\frac{f(x)-f(a)}{x-a} - f'(a)\right| + \left|\frac{g(x)-g(a)}{x-a} - g'(a)\right|
\end{aligned}
$$

Both $f$ and $g$ meet the requirements of Cauchy's definition. Whatever the value of our given $\epsilon$, we can let $\epsilon_1 = \frac{\epsilon}{2}$.

$$\begin{aligned}
|E(x,a)| &= \left|\frac{F(x)-F(a)}{x-a} - (f'(a)+g'(a))\right| \\
&= \left|\frac{f(x)+g(x)-f(a)-g(a)}{x-a} - (f'(a)+g'(a))\right| \\
&= \left|\frac{f(x)-f(a)}{x-a} - f'(a) + \frac{g(x)-g(a)}{x-a} - g'(a)\right| \\
&\leq \left|\frac{f(x)-f(a)}{x-a} - f'(a)\right| + \left|\frac{g(x)-g(a)}{x-a} - g'(a)\right|
\end{aligned}$$

Both $f$ and $g$ meet the requirements of Cauchy's definition. Whatever the value of our given $\epsilon$, we can let $\epsilon_1 = \frac{\epsilon}{2}$. Apply Cauchy's definition to $f$ and $g$ separately using the number $\epsilon_1$ to find a $\delta_f$ that works for $f$ and a possibly different $\delta_g$ that works for $g$.

$$
\begin{aligned}
|E(x,a)| &= \left|\frac{F(x)-F(a)}{x-a} - (f'(a)+g'(a))\right| \\
&= \left|\frac{f(x)+g(x)-f(a)-g(a)}{x-a} - (f'(a)+g'(a))\right| \\
&= \left|\frac{f(x)-f(a)}{x-a} - f'(a) + \frac{g(x)-g(a)}{x-a} - g'(a)\right| \\
&\leq \left|\frac{f(x)-f(a)}{x-a} - f'(a)\right| + \left|\frac{g(x)-g(a)}{x-a} - g'(a)\right|
\end{aligned}
$$

Both $f$ and $g$ meet the requirements of Cauchy's definition. Whatever the value of our given $\epsilon$, we can let $\epsilon_1 = \frac{\epsilon}{2}$. Apply Cauchy's definition to $f$ and $g$ separately using the number $\epsilon_1$ to find a $\delta_f$ that works for $f$ and a possibly different $\delta_g$ that works for $g$.Then let $\delta = min(\delta_f, \delta_g)$.

$$\begin{aligned}|E(x,a)| &= \left|\frac{F(x)-F(a)}{x-a} - (f'(a)+g'(a))\right| \\ &= \left|\frac{f(x)+g(x)-f(a)-g(a)}{x-a} - (f'(a)+g'(a))\right| \\ &= \left|\frac{f(x)-f(a)}{x-a} - f'(a) + \frac{g(x)-g(a)}{x-a} - g'(a)\right| \\ &\leq \left|\frac{f(x)-f(a)}{x-a} - f'(a)\right| + \left|\frac{g(x)-g(a)}{x-a} - g'(a)\right|\end{aligned}$$

Both $f$ and $g$ meet the requirements of Cauchy's definition. Whatever the value of our given $\epsilon$, we can let $\epsilon_1 = \frac{\epsilon}{2}$. Apply Cauchy's definition to $f$ and $g$ separately using the number $\epsilon_1$ to find a $\delta_f$ that works for $f$ and a possibly different $\delta_g$ that works for $g$.Then let $\delta = min(\delta_f, \delta_g)$. If $0 < |x-a| < \delta$ then the above shows that

$$E(x,a) < \epsilon_1 + \epsilon_1 = \epsilon$$

For a function defined by a series

$$F(x) = f_1(x) + f_2(x) + f_3(x) \cdots = \sum_{k=1}^{\infty} f_k(x)$$

where would the proof above break down?

For a function defined by a series

$$F(x) = f_1(x) + f_2(x) + f_3(x) \cdots = \sum_{k=1}^{\infty} f_k(x)$$

where would the proof above break down?

That is, if $f_1, f_2, \cdots$ all have derivatives at point $a$ and we let

$$E(x, a) \;=\; \frac{\sum_{k=1}^{\infty} f_k(x) - \sum_{k=1}^{\infty} f_k(a)}{x - a} - \sum_{k=1}^{\infty} f_k'(a)$$

For a function defined by a series

$$F(x) = f_1(x) + f_2(x) + f_3(x) \cdots = \sum_{k=1}^{\infty} f_k(x)$$

where would the proof above break down?

That is, if $f_1, f_2, \cdots$ all have derivatives at point $a$ and we let

$$\begin{aligned} E(x,a) &= \frac{\sum_{k=1}^{\infty} f_k(x) - \sum_{k=1}^{\infty} f_k(a)}{x-a} - \sum_{k=1}^{\infty} f_k'(a) \\ &= \sum_{k=1}^{\infty} \left(\frac{f_k(x) - f_k(a)}{x-a} - f_k'(a)\right) \end{aligned}$$

For a function defined by a series

$$F(x) = f_1(x) + f_2(x) + f_3(x) \cdots = \sum_{k=1}^{\infty} f_k(x)$$

where would the proof above break down?

That is, if $f_1, f_2, \cdots$ all have derivatives at point $a$ and we let

$$\begin{aligned} E(x,a) &= \frac{\sum_{k=1}^{\infty} f_k(x) - \sum_{k=1}^{\infty} f_k(a)}{x-a} - \sum_{k=1}^{\infty} f_k'(a) \\ &= \sum_{k=1}^{\infty} \left( \frac{f_k(x) - f_k(a)}{x-a} - f_k'(a) \right) \end{aligned}$$

What could go wrong?

Whether or not a functions is *continuous* deserves considerable discussion in later sections.

Whether or not a functions is *continuous* deserves considerable discussion in later sections.

More about that to come, but for now, I think it is safe to assume that everyone has at least a good intuitive idea of what it means for a function to be continuous:

Whether or not a functions is *continuous* deserves considerable discussion in later sections.

More about that to come, but for now, I think it is safe to assume that everyone has at least a good intuitive idea of what it means for a function to be continuous: We will need to spend some time time with the Mean Value Theorem (!) in upcoming sections and chapters.

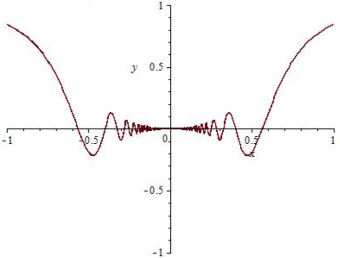
Whether or not a functions is *continuous* deserves considerable discussion in later sections.

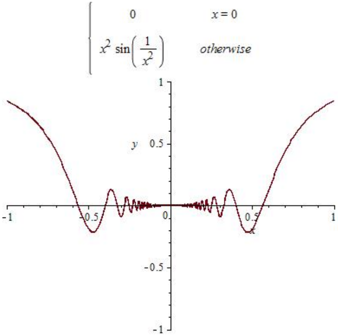
More about that to come, but for now, I think it is safe to assume that everyone has at least a good intuitive idea of what it means for a function to be continuous: We will need to spend some time time with the Mean Value Theorem (!) in upcoming sections and chapters. Cauchy's proof of the Mean Value Theorem made an assumption that the derivative is continuous.

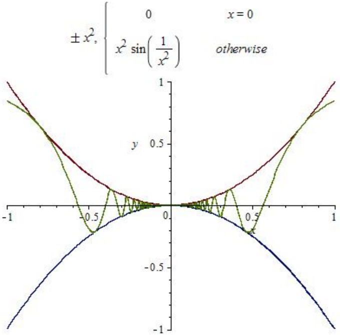
But consider the following function:

$$f(x) = \begin{cases} x^2 \sin(\frac{1}{x^2}) & : x \neq 0 \\ 0 & : x = 0 \end{cases}$$

$$\begin{cases} 0 & x = 0 \\ x^2 \sin\left(\frac{1}{x^2}\right) & \textit{otherwise} \end{cases}$$






The graph of f is squeezed between $x^2$ and $-x^2$

It is not difficult to show that the derivative of $f$ exists everywhere, including at $x = 0$, and that

$$f'(x) = \begin{cases} 2x\sin(\frac{1}{x^2}) - 2\frac{cos(\frac{1}{x^2})}{x}; & : x \neq 0 \\ 0 & : x = 0 \end{cases}$$

It is not difficult to show that the derivative of $f$ exists everywhere, including at $x = 0$, and that

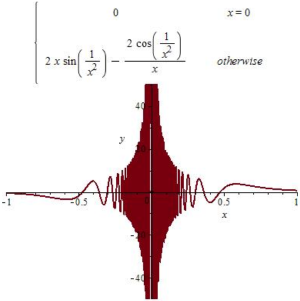
$$f'(x) = \begin{cases} 2x\sin(\frac{1}{x^2}) - 2\frac{cos(\frac{1}{x^2})}{x}; & : x \neq 0 \\ 0 & : x = 0 \end{cases}$$

One way to think about the derivative at $x = 0$ is the fact that $f$ is squeezed between the two differentiable functions $\pm x^2$.

It is not difficult to show that the derivative of $f$ exists everywhere, including at $x = 0$, and that

$$f'(x) = \begin{cases} 2x\sin(\frac{1}{x^2}) - 2\frac{\cos(\frac{1}{x^2})}{x}; & : x \neq 0 \\ 0 & : x = 0 \end{cases}$$

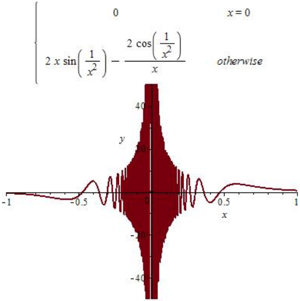
One way to think about the derivative at $x = 0$ is the fact that $f$ is squeezed between the two differentiable functions $\pm x^2$. But the derivative is **not** continuous at $x = 0$.



It is not difficult to show that the derivative of $f$ exists everywhere, including at $x = 0$, and that

$$f'(x) = \begin{cases} 2x \sin(\frac{1}{x^2}) - 2\frac{cos(\frac{1}{x^2})}{x}; & : x \neq 0 \\ 0 & : x = 0 \end{cases}$$

One way to think about the derivative at $x = 0$ is the fact that $f$ is squeezed between the two differentiable functions $\pm x^2$. But the derivative is **not** continuous at $x = 0$.

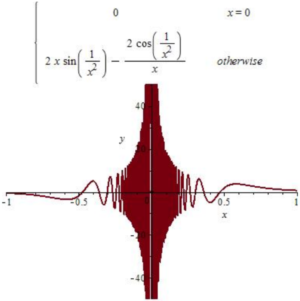



So ... Cauchy's proof?

It is not difficult to show that the derivative of $f$ exists everywhere, including at $x = 0$, and that

$$f'(x) = \begin{cases} 2x\sin(\frac{1}{x^2}) - 2\frac{cos(\frac{1}{x^2})}{x}; & : x \neq 0 \\ 0 & : x = 0 \end{cases}$$

One way to think about the derivative at $x = 0$ is the fact that $f$ is squeezed between the two differentiable functions $\pm x^2$. But the derivative is **not** continuous at $x = 0$.




So ... Cauchy's proof? Not so much...

Another interesting function:

$$g(x) = \begin{cases} e^{-\frac{1}{x^2}} & : x \neq 0 \\ 0 & : x = 0 \end{cases}$$

$g'(x)$ exists at all values of $x$, including $x = 0$

$g'(x)$ exists at all values of $x$, including $x = 0$.

In fact, the same is true for $g'$, $g''$, $g'''$, etc.

$g'(x)$ exists at all values of $x$, including $x = 0$.

In fact, the same is true for $g'$, $g''$, $g'''$, etc.

Here are the formulas for $g'$ and $g''$:

$$g'(x) = \begin{cases} e^{-\frac{1}{x^2}} \cdot \frac{2}{x^3} & : x \neq 0 \\ 0 & : x = 0 \end{cases}$$

$g'(x)$ exists at all values of $x$, including $x = 0$.

In fact, the same is true for $g'$, $g''$, $g'''$, etc.

Here are the formulas for $g'$ and $g''$:

$$g'(x) = \begin{cases} e^{-\frac{1}{x^2}} \cdot \frac{2}{x^3} & : x \neq 0 \\ 0 & : x = 0 \end{cases}$$

$$g''(x) = \begin{cases} e^{-\frac{1}{x^2}} \left(-\frac{6}{x^4} + \frac{4}{x^6}\right) & : x \neq 0 \\ 0 & : x = 0 \end{cases}$$

$g'(x)$ exists at all values of $x$, including $x = 0$.

In fact, the same is true for $g'$, $g''$, $g'''$, etc.

Here are the formulas for $g'$ and $g''$:

$$g'(x) = \begin{cases} e^{-\frac{1}{x^2}} \cdot \frac{2}{x^3} & : x \neq 0 \\ 0 & : x = 0 \end{cases}$$

$$g''(x) = \begin{cases} e^{-\frac{1}{x^2}} (-\frac{6}{x^4} + \frac{4}{x^6}) & : x \neq 0 \\ 0 & : x = 0 \end{cases}$$

And in addition $g(0) = g'(0) = g''(0) = \cdots = g^{(n)}(0) = \cdots = 0$.

$g'(x)$ exists at all values of $x$, including $x = 0$.

In fact, the same is true for $g'$, $g''$, $g'''$, etc.

Here are the formulas for $g'$ and $g''$:

$$g'(x) = \begin{cases} e^{-\frac{1}{x^2}} \cdot \frac{2}{x^3} & : x \neq 0 \\ 0 & : x = 0 \end{cases}$$

$$g''(x) = \begin{cases} e^{-\frac{1}{x^2}} (-\frac{6}{x^4} + \frac{4}{x^6}) & : x \neq 0 \\ 0 & : x = 0 \end{cases}$$

And in addition $g(0) = g'(0) = g''(0) = \cdots = g^{(n)}(0) = \cdots = 0$. Give some thought as to why this is significant, especially as relating to Taylor's theorem.