MATH 510, Notes 5

Modern Analysis

James Madison University

Mean Value Theorem (Taylor's Theorem with $n = 1$): If f is differentiable at all points strictly between a and b and continuous at a and b and all points in between, then there is a number c strictly between a and b so that

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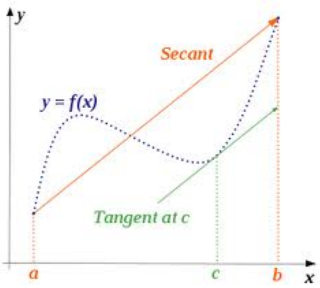

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"Continuous on set A" $\Leftrightarrow$ "Continuous at every point in set A."

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It might appear that IVP actually characterizes continuous functions, but it may be that other non-continuous functions have this property. Can you think of any?

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Given an increasing sequence $x_1 \leq x_2 \leq x_3 \leq \cdots$ and a decreasing sequence $y_1 \geq y_3 \geq y_3 \geq \cdots$ such that y_n is always larger than x_n but the difference between y_n and x_n can be made arbitrarily small by taking n sufficiently large, there is **exactly one** real number that is greater than or equal to every x_n and less than or equal to every y_n.

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Algebraic numbers are those that are solutions to polynomial equations with rational co-efficients, which of course includes the solutions to $x^2 - 2 = 0$. π is not the solution to any such equation.

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Can you do that?

Some other things to think about regarding the real numbers. Maybe you are familiar with this and maybe not:

- The set of rational numbers is described as *countable* because it is possible to write the rational numbers in a sequence. This is equivalent to saying that there is a one to one correspondence between the natural numbers and the rational numbers.

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Can you see how to prove that? Also note: Once we see there is at least one rational between any two real numbers, it quickly becomes "obvious" that there are an infinite number.(Maybe "obvious" is a bit strong, but can you see that this is true?)

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In the above, we are assuming that the rational number x is written as the ratio of two integers p and q with the fraction in reduced form.

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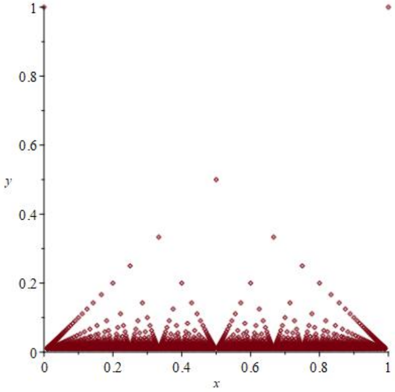

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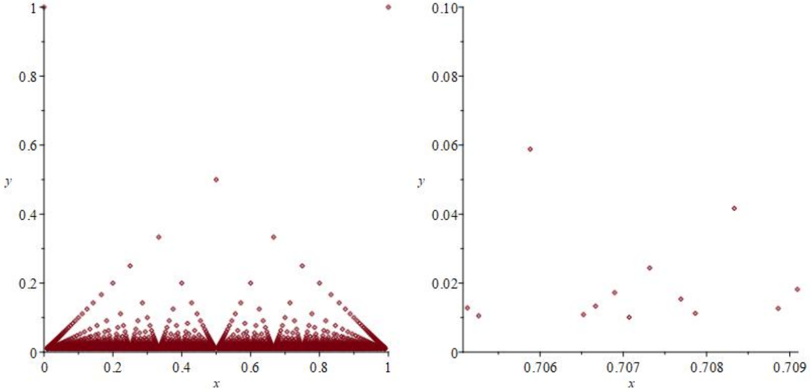

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Finally, the following theorem:

Theorem If f is differentiable at $x = c$ (i.e. $f'(c)$ exists) then f is continous at $x = c$.

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The proof depends on the completeness of the real numbers, using the nested interval principle.Superficially, it might appear that the theorem could still be stated and make sense using just the rational numbers. What is missing in that interpretation, or is it the case that the completeness of the real numbers is no essential for the theorem?

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Similarly: bounded below, greatest lower bound, infimum...

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This is an equivalent way to describe the completeness of the real numbers. The nested interval principle may be used to prove the above, or alternatively if we began with the above then we could prove the nested interval principle.

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Extremum - this is another term for what is also referred to as a *local* maximum or minimum. f has an extremum at point c if $f(x) \leq f(c)$ (or alternatively $f(x) \geq f(c)$) for all x "near" point c. That is, for all x in some open interval or "neighborhood" containing c, no matter how small that neighborhood might be.

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For example, the function f defined by $f(x) = \sin(\frac{1}{x}$ has an infinite number of extrema between 0 and 1.

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And using the above and some of the deeper preceding results about maximum and minimum values:

Rolle's Theorem Suppose that function g is continuous on a closed interval $[a, b]$ and differentiable at least on the open interval (a, b) and in addition $g(a) = g(b)$, then there is at least one point c in (a, b) such that $f'(c) = 0$. .

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Can you give a two line explanation of why Rolle's Theorem must be true?

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proof: If f has the properties described above, then define a new function g as follows:

$$g(x)=f(x)-f(a)-\frac{f(b)-f(a)}{b-a}(x-a)$$

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Theorem Suppose that f and F are functions that are continuous on a closed interval $[a, b]$ and differentiable at least on the open interval (a, b) and in addition $F'(x)$ is never equal to zero between a and b. Then there is a number c strictly between a and b so that

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$$g(x) = F(x)(f(b) - f(a)) - f(x)(F(b) - F(a))$$

and think about $g(a)$, $g(b)$, and the formula for $g'(x)$.

Taylor's Theorem Suppose that the domain of f is an open interval, a is a number in the domain, and that f has derivatives (at least through the $n-$th). If x is another number in the domain then there is a number c between a and x so that:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 +$$

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$\frac{f^{(n)}(c)}{n!}(x-a)^n$ is call the *Lagrange Remainder*.
There is also a similar theorem with the *Cauchy Remainder*:
$\frac{f^{(n)}(c)}{(n-1)!}(x-c)^{n-1}(x-a)$

Another consequence of the MVT:

Theorem Suppose that f and F are functions that are differentiable on an open interval that contains point a. In addition

$$\lim_{x \to a} f(x) = 0 = \lim_{x \to a} F(x).$$

If $F'(x)$ is never equal to zero in this open interval and if $\lim_{x \to a} \frac{f'(x)}{F'(x)}$ exists, then

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There are several other versions of this theorem, involving limts as $x- > \infty$ instead of $x- > a$ or involving an "∞/∞" form.