MATH 510, Notes 5

Modern Analysis

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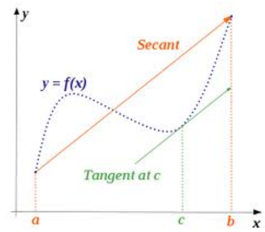
Mean Value Theorem (Taylor's Theorem with $n = 1$): If f is differentiable at all points strictly between a and b and continuous at a and b and all points in between, then there is a number c strictly between a and b so that

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$



Continuity was mentioned above, without a definition. In the following, we assume that point a is in the domain of f.

Function f is **continuous** at point a if given any positive *error tolerance* ϵ it is possible to find a corresponding positive input tolerance δ for the domain so that

$$|x - a| < \delta$$

implies that

$$|f(x) - f(a)| < \epsilon$$

Shorthand altenative: Function f is **continuous** at point a if

$$\lim_{x \to a} f(x) = f(a)$$

"Continuous on set A" $\Leftrightarrow$ "Continuous at every point in set A."

Theorem If f is continuous on the interval $[x_1, x_2]$ and α is a number between $f(x_1)$ and $f(x_2)$ then there is a number c between x_1 and x_2 so that $f(c) = \alpha$.

That is, if f is continuous, then f has the so-called *intermediate value property*:

Function f has the IVP on an interval $[a, b]$ if for any pair of points x_1 and x_2 in $[a, b]$ α is a number between $f(x_1)$ and $f(x_2)$ then there is a number c between x_1 and x_2 so that $f(c) = \alpha$.

It might appear that IVP actually characterizes continuous functions, but it may be that other non-continuous functions have this property. Can you think of any?

Proving the IVT is sort-of a big deal - this is quite definitely not quick and easy:
The text provides a proof based on the *Nested Interval Principle:*

Given an increasing sequence $x_1 \leq x_2 \leq x_3 \leq \cdots$ and a decreasing sequence $y_1 \geq y_3 \geq y_3 \geq \cdots$ such that y_n is always larger than x_n but the difference between y_n and x_n can be made arbitrarily small by taking n sufficiently large, there is **exactly one** real number that is greater than or equal to every x_n and less than or equal to every y_n.

The Nested Interval Principle is equivalent to the Least Upper Bound (or Greatest Lower Bound) property, and both equivalent to the so-called Completeness property of the Real Numbers. (See page 125?) And also the conclusion of the Bolzano Weierstrauss Theorem. You can look these up if they are not familiar, although we will discuss them in more detail later.

We discuss properties of the rational numbers and the real numbers in school, but "standards" do not often mention this fundamental concept that characterizes the real number system. Students are somewhat aware that in a sense there are "more" real numbers. Generally they can name some real numbers that are not rational numbers: $\sqrt{2}$? π?

$\sqrt{2}$ is an example of an *algebraic* number, a set of numbers that includes the rationals but in a sense does not go too much beyond the rationals. π is an example of a *trancendental* number, a real number which is not algebraic.
Algebraic numbers are those that are solutions to polynomial equations with rational co-efficients, which of course includes the solutions to $x^2 - 2 = 0$. π is not the solution to any such equation.

Some things you should be able to do:

- Describe an example that would show that the rational numbers do not have the nested interval property.
- Show that $\sqrt{2}$ is not a rational number.

Can you do that?

Some other things to think about regarding the real numbers. Maybe you are familiar with this and maybe not:

- The set of rational numbers is described as *countable* because it is possible to write the rational numbers in a sequence. This is equivalent to saying that there is a one to one correspondence between the natural numbers and the rational numbers.
- The set of algebraic numbers is also countable.
- The set of real numbers is **not** countable. It is **not** possible to make a one to one correspondence between the real numbers and the natural numbers.

So, in some sense, the natural numbers, rational numbers, and algebraic numbers all have the same "size." The set of real numbers is "larger."
For the record, symbols: Real numbers = $\mathbb{R}$, rational numbers = $\mathbb{Q}$, natural numbers = $\mathbb{N}$, integers = $\mathbb{Z}$ (Why Z?)

Many interesting things are a consequence of the fact that $\mathbb{Q}$ is *dense* in $\mathbb{R}$. That is:

Between any two real numbers (no matter how close together) it is possible to find a rational number.

Can you see how to prove that? Also note: Once we see there is at least one rational between any two real numbers, it quickly becomes "obvious" that there are an infinite number.(Maybe "obvious" is a bit strong, but can you see that this is true?)

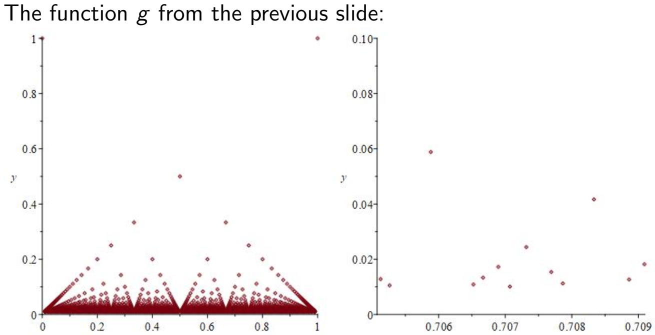
Some additional interesting functions to think about:

$$f(x) = \begin{cases} x & : x \in \mathbb{Q} \\ 0 & : x \notin \mathbb{Q} \end{cases}$$

$$g(x) = \begin{cases} 1 & : x = 0 \\ 1/q & : x \in \mathbb{Q}, x = p/q \\ 0 & : x \notin \mathbb{Q} \end{cases}$$

In the above, we are assuming that the rational number x is written as the ratio of two integers p and q with the fraction in reduced form.

The function g from the previous slide:



Be sure that you could answer and explain the following:

- For which values of x is the function f on the previous slide continuous?
- For which values of x is the function g on the previous slide continuous?

We are not going to do an extensive discussion of proving that typical combinations of continuous functions through addition, multiplication, division (if the denominator is $\neq 0$), and composition are also continuous. Typical proofs: pages 88-90. These should make sense, and you should be able to reproduce most of them if you had to(!).

Finally, the following theorem:

Theorem If f is differentiable at $x = c$ (i.e. $f'(c)$ exists) then f is continous at $x = c$.

The converse is not true: Example?

Theorem If function f is continuous on the interval $[a, b]$, then f is *bounded*. That is, there are numbers A and B so that

$$A \leq f(x) \leq B$$

for all $x \in [a, b]$.

Is it necessary that the theorem refers to a *closed* interval?

The proof depends on the completeness of the real numbers, using the nested interval principle.Superficially, it might appear that the theorem could still be stated and make sense using just the rational numbers. What is missing in that interpretation, or is it the case that the completeness of the real numbers is no essential for the theorem?

Number B is an upper bound for set $\mathbf{S}$ if $s \leq B$ for all $s \in \mathbf{S}$. We then may say that $\mathbf{S}$ is bounded above.

If smallest upper bound for a set that is bounded above is the *least upper bound*, sometimes referred to as the *supremum*. I.e. the supremum for a set is greater than or equal to every number in the set, but less than or equal to any upper bound for the set.

Similarly: bounded below, greatest lower bound, infimum...

Every set of real numbers that is bounded above has a least upper bound. Every set of real numbers that is bounded below has a greatest lower bound.

This is an equivalent way to describe the completeness of the real numbers. The nested interval principle may be used to prove the above, or alternatively if we began with the above then we could prove the nested interval principle.

Theorem If f is continuous on the interval $[a, b]$ then there are numbers $k_1, k_2 \in [a, b]$ so that

$$f(k_1) \leq f(x) \leq f(k_2).$$

You should be able to give examples to show that the conclusion of the theorem is not necessarily true if the premises are not met. That is, if the interval is open.Or infinite. Or if the function is not continuous.

Extremum - this is another term for what is also referred to as a *local* maximum or minimum. f has an extremum at point c if $f(x) \le f(c)$ (or alternatively $f(x) \ge f(c)$) for all x "near" point c. That is, for all x in some open interval or "neighborhood" containing c, no matter how small that neighborhood might be.

For example, the function f defined by $f(x) = \sin(\frac{1}{x}$ has an infinite number of extrema between 0 and 1.

A theorem that is not so difficult to prove...
Theorem If function f has an extremum at point c and f is differentiable in an open interval containing c, then $f'(c) = 0$.

And using the above and some of the deeper preceding results about maximum and minimum values:
Rolle's Theorem Suppose that function g is continuous on a closed interval $[a, b]$ and differentiable at least on the open interval (a, b) and in addition $g(a) = g(b)$, then there is at least one point c in (a, b) such that $f'(c) = 0$. .

Can you give a two line explanation of why Rolle's Theorem must be true?

Flashback:

MVT If f is differentiable at all points strictly between a and b and continuous at a and b and all points in between, then there is a number c strictly between a and b so that

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

proof: If f has the properties described above, then define a new function g as follows:

$$g(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a)$$

Then $g(a) =?$ $g(b) =?$

Note that:

$$g'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}$$

The so-called "Generalized MVT":

Theorem Suppose that f and F are functions that are continuous on a closed interval $[a, b]$ and differentiable at least on the open interval (a, b) and in addition $F'(x)$ is never equal to zero between a and b. Then there is a number c strictly between a and b so that

$$\frac{f(b) - f(a)}{F(b) - F(a)} = \frac{f'(c)}{F'(c)}$$

Mysterious as this seems, the proof is pretty easy: let

$$g(x) = F(x)(f(b) - f(a)) - f(x)(F(b) - F(a))$$

and think about $g(a)$, $g(b)$, and the formula for $g'(x)$.

Taylor's Theorem Suppose that the domain of f is an open interval, a is a number in the domain, and that f has derivatives (at least through the $n-$th). If x is another number in the domain then there is a number c between a and x so that:

$$f(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \frac{f'''(a)}{3!}(x - a)^3 +$$

$$\cdots + \frac{f^{(n-1)}(a)}{(n-1)!}(x - a)^{n-1} + \frac{f^{(n)}(c)}{n!}(x - a)^n$$

We are not going to outline a proof here, but the proof requires repeated application of the Generalized MVT.
$\frac{f^{(n)}(c)}{n!}(x - a)^n$ is call the *Lagrange Remainder*.
There is also a similar theorem with the *Cauchy Remainder*:
$\frac{f^{(n)}(c)}{(n-1)!}(x - c)^{n-1}(x - a)$

Another consequence of the MVT:
Theorem Suppose that f and F are functions that are differentiable on an open interval that contains point a. In addition

$$\lim_{x \to a} f(x) = 0 = \lim_{x \to a} F(x).$$

If $F'(x)$ is never equal to zero in this open interval and if $\lim_{x \to a} \frac{f'(x)}{F'(x)}$ exists, then

$$\lim_{x \to a} \frac{f(x)}{F(x)} = \lim_{x \to a} \frac{f'(x)}{F'(x)}.$$

The above could be referred to as the "0/0" form of L'Hôpital's rule, although of course "0/0" is a meaningless expression and merely shorthand for the limit property above.
There are several other versions of this theorem, involving limts as $x- > \infty$ instead of $x- > a$ or involving an "∞/∞" form.