MATH 510, Notes 6

Modern Analysis

James Madison University

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The proof depends on the completeness of the real numbers, using the nested interval principle.Superficially, it might appear that the theorem could still be stated and make sense using just the rational numbers. What is missing in that interpretation, or is it the case that the completeness of the real numbers is no essential for the theorem?

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Similarly: bounded below, greatest lower bound, infimum...

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This is an equivalent way to describe the completeness of the real numbers. The nested interval principle may be used to prove the above, or alternatively if we began with the above then we could prove the nested interval principle.

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You should be able to give examples to show that the conclusion of the theorem is not necessarily true if the premises are not met. That is, if the interval is open.Or infinite. Or if the function is not continuous.

Extremum - this is another term for what is also referred to as a *local* maximum or minimum. f has an extremum at point c if $f(x) \leq f(c)$ (or alternatively $f(x) \geq f(c)$) for all x "near" point c. That is, for all x in some open interval or "neighborhood" containing c, no matter how small that neighborhood might be.

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For example, the function f defined by $f(x) = \sin(\frac{1}{x}$ has an infinite number of extrema between 0 and 1.

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And using the above and some of the deeper preceding results about maximum and minimum values:

Rolle's Theorem Suppose that function g is continuous on a closed interval $[a, b]$ and differentiable at least on the open interval (a, b) and in addition $g(a) = g(b)$, then there is at least one point c in (a, b) such that $f'(c) = 0$. .

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Can you give a two line explanation of why Rolle's Theorem must be true?

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MVT If f is differentiable at all points strictly between a and b and continuous at a and b and all points in between, then there is a number c strictly between a and b so that

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

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proof: If f has the properties described above, then define a new function g as follows:

$$g(x)=f(x)-f(a)-\frac{f(b)-f(a)}{b-a}(x-a)$$

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The so-called "Generalized MVT":

Theorem Suppose that f and F are functions that are continuous on a closed interval $[a, b]$ and differentiable at least on the open interval (a, b) and in addition $F'(x)$ is never equal to zero between a and b. Then there is a number c strictly between a and b so that

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Mysterious as this seems, the proof is pretty easy: let

$$g(x) = F(x)(f(b) - f(a)) - f(x)(F(b) - F(a))$$

and think about $g(a)$, $g(b)$, and the formula for $g'(x)$.

Taylor's Theorem Suppose that the domain of f is an open interval, a is a number in the domain, and that f has derivatives (at least through the $n-$th). If x is another number in the domain then there is a number c between a and x so that:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n-1)}(a)}{(n-1)!}(x-a)^{n-1} + \frac{f^{(n)}(c)}{n!}(x-a)^n$$

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$\frac{f^{(n)}(c)}{n!}(x-a)^n$ is call the *Lagrange Remainder*.
There is also a similar theorem with the *Cauchy Remainder*:
$\frac{f^{(n)}(c)}{(n-1)!}(x-c)^{n-1}(x-a)$

Another consequence of the MVT:

Theorem Suppose that f and F are functions that are differentiable on an open interval that contains point a. In addition

$$\lim_{x \to a} f(x) = 0 = \lim_{x \to a} F(x).$$

If $F'(x)$ is never equal to zero in this open interval and if $\lim_{x \to a} \frac{f'(x)}{F'(x)}$ exists, then

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There are several other versions of this theorem, involving limts as $x->\infty$ instead of $x->a$ or involving an "∞/∞" form.