MATH 510, Notes 6

Modern Analysis

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Theorem If function f is continuous on the interval $[a, b]$, then f is *bounded*. That is, there are numbers A and B so that

$$A \leq f(x) \leq B$$

for all $x \in [a, b]$.

Is it necessary that the theorem refers to a *closed* interval?

The proof depends on the completeness of the real numbers, using the nested interval principle.Superficially, it might appear that the theorem could still be stated and make sense using just the rational numbers. What is missing in that interpretation, or is it the case that the completeness of the real numbers is no essential for the theorem?

Number B is an upper bound for set $\mathbf{S}$ if $s \leq B$ for all $s \in \mathbf{S}$. We then may say that $\mathbf{S}$ is bounded above.

If smallest upper bound for a set that is bounded above is the *least upper bound*, sometimes referred to as the *supremum*. I.e. the supremum for a set is greater than or equal to every number in the set, but less than or equal to any upper bound for the set.

Similarly: bounded below, greatest lower bound, infimum...

Every set of real numbers that is bounded above has a least upper bound. Every set of real numbers that is bounded below has a greatest lower bound.

This is an equivalent way to describe the completeness of the real numbers. The nested interval principle may be used to prove the above, or alternatively if we began with the above then we could prove the nested interval principle.

Theorem If f is continuous on the interval $[a, b]$ then there are numbers $k_1, k_2 \in [a, b]$ so that

$$f(k_1) \leq f(x) \leq f(k_2).$$

You should be able to give examples to show that the conclusion of the theorem is not necessarily true if the premises are not met. That is, if the interval is open.Or infinite. Or if the function is not continuous.

Extremum - this is another term for what is also referred to as a *local* maximum or minimum. f has an extremum at point c if $f(x) \leq f(c)$ (or alternatively $f(x) \geq f(c)$) for all x "near" point c. That is, for all x in some open interval or "neighborhood" containing c, no matter how small that neighborhood might be.

For example, the function f defined by $f(x) = \sin(\frac{1}{x}$ has an infinite number of extrema between 0 and 1.

A theorem that is not so difficult to prove...
Theorem If function f has an extremum at point c and f is differentiable in an open interval containing c, then $f'(c) = 0$.

And using the above and some of the deeper preceding results about maximum and minimum values:
Rolle's Theorem Suppose that function g is continuous on a closed interval $[a, b]$ and differentiable at least on the open interval (a, b) and in addition $g(a) = g(b)$, then there is at least one point c in (a, b) such that $f'(c) = 0$. .

Can you give a two line explanation of why Rolle's Theorem must be true?

Flashback:

MVT If f is differentiable at all points strictly between a and b and continuous at a and b and all points in between, then there is a number c strictly between a and b so that

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

proof: If f has the properties described above, then define a new function g as follows:

$$g(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a)$$

Then $g(a) =?$ $g(b) =?$

Note that:

$$g'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}$$

The so-called "Generalized MVT":

Theorem Suppose that f and F are functions that are continuous on a closed interval $[a, b]$ and differentiable at least on the open interval (a, b) and in addition $F'(x)$ is never equal to zero between a and b. Then there is a number c strictly between a and b so that

$$\frac{f(b) - f(a)}{F(b) - F(a)} = \frac{f'(c)}{F'(c)}$$

Mysterious as this seems, the proof is pretty easy: let

$$g(x) = F(x)(f(b) - f(a)) - f(x)(F(b) - F(a))$$

and think about $g(a)$, $g(b)$, and the formula for $g'(x)$.

Taylor's Theorem Suppose that the domain of f is an open interval, a is a number in the domain, and that f has derivatives (at least through the n−th). If x is another number in the domain then there is a number c between a and x so that:

$$f(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \frac{f'''(a)}{3!}(x - a)^3 +$$

$$\cdots + \frac{f^{(n-1)}(a)}{(n-1)!}(x - a)^{n-1} + \frac{f^{(n)}(c)}{n!}(x - a)^n$$

We are not going to outline a proof here, but the proof requires repeated application of the Generalized MVT.
$\frac{f^{(n)}(c)}{n!}(x - a)^n$ is call the *Lagrange Remainder*.
There is also a similar theorem with the *Cauchy Remainder*:
$\frac{f^{(n)}(c)}{(n-1)!}(x - c)^{n-1}(x - a)$

Another consequence of the MVT:
Theorem Suppose that f and F are functions that are differentiable on an open interval that contains point a. In addition

$$\lim_{x \to a} f(x) = 0 = \lim_{x \to a} F(x).$$

If $F'(x)$ is never equal to zero in this open interval and if $\lim_{x \to a} \frac{f'(x)}{F'(x)}$ exists, then

$$\lim_{x \to a} \frac{f(x)}{F(x)} = \lim_{x \to a} \frac{f'(x)}{F'(x)}.$$

The above could be referred to as the "0/0" form of L'Hôpital's rule, although of course "0/0" is a meaningless expression and merely shorthand for the limit property above.
There are several other versions of this theorem, involving limts as $x -> \infty$ instead of $x -> a$ or involving an "∞/∞" form.