# MATH 510, Notes 7

Modern Analysis

James Madison University

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$a_1, a_2, \cdots$ - *summands*

Saying that the series converges to $T$,

$$\sum_{k=1}^{\infty} a_k = T$$

means that the sequence of partial sums converges to $T$. (Given $\epsilon > 0$ there exists $N$ so that ... etc.)

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Sometimes mixing up the terms of a series changes the target/limit, and sometimes not.

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But of course that only tells us when a series *might* converge.

Or, more often, that might tell us when a series does *not* converge.

Preview: We do not want to bypass the most straightforward sort of test about convergent or divergent series. The author does not mention until section 4.2, but let's put it out there now:

**Theorem** Given two infinite series $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$, suppose that for all values of $k$ we have

$$0 \leq a_k \leq b_k.$$

If the series $\sum_{k=1}^{\infty} b_k$ converges, then $\sum_{k=1}^{\infty} a_k$ does also. If the series $\sum_{k=1}^{\infty} a_k$ diverges, then $\sum_{k=1}^{\infty} b_k$ does also.

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Can you see why theorem is still true if instead we know $0 \leq a_k \leq C \cdot b_k$ for some positive constant $C$? What if we only know that $0 \leq a_k \leq b_k$ "eventually?" That is, after some finite number of terms at the beginning of the series.

**Definition** An infinite sequence $\{S_1, S_2, ...\}$ is *Cauchy* if given $\epsilon > 0$ we can find a number $N$ so that for any numbers $n$ and $m$ with $n > m \geq N$ we have

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A series is *Cauchy* when its sequence of partial sums is Cauchy.

**Theorem** A sequence or series converges iff it is Cauchy.

To prove this, we need to think about the following ideas:
Given a *bounded* sequence $\{S_n\}$ let:
$U_1 = l.u.b\{S_1, S_2, S_3, \cdots\}$
$U_2 = l.u.b\{S_2, S_3, S_4, \cdots\}$
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In a similar way define a sequence $\{L_n\}$ using g.l.b. instead of l.u.b.

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Similarly there is a limit inferior $\liminf_n S_n$ defined using $\{L_n\}$

(Note: the idea of *limsup* and *liminf* may be extended to unbounded sequences if we allow *limsup* and *liminf* to take on the values $+\infty$ or $-\infty$.)

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One problem is that we do not know up front what should be the value for the target/limit. But we can get this value either using the nested interval principle applied to the intervals $[L_n, U_n]$, or more directly by showing that for a Cauchy sequence the limsup is the same as the liminf, and that this number is also the limit.

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If a series $\sum_{k=1}^{\infty} a_k$ converges to $T$, can we tell what $\sum_{k=1}^{\infty} |a_k|$ converges to?

**Theorem** If a series converges absolutely then the series converges in the usual sense. That is, if

$$\sum_{k=1}^{\infty} |a_k|$$

converges, then

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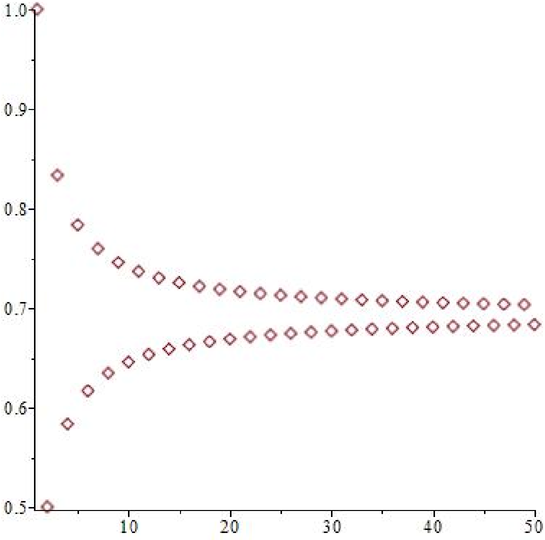

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**If** a series converges conditionally, it must be the case that separately both $\sum_{k=1}^{\infty} a_k^+$ and $\sum_{k=1}^{\infty} a_k^-$ diverge, but somehow their interaction allows them to converge when the partial sums are subtracted.

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So we will first concentrate on the absolute convergence, where a great deal can be done with so-called "comparison tests" and related tests that indirectly depend on comparisons.

In much of what follows, the theorems will assume that we have non-negative summands. That is, series $\sum_{k=1}^{\infty} a_k$ for which all $a_k \geq 0$.

But of course these theorems would apply to other not necessarily non-negative (is that a triple negative?!?) series if we are testing for absolute convergence by applying the theorems to $\sum_{k=1}^{\infty} |a_k|$.

Recall: **Theorem** Given two series $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ with $0 \leq a_n \leq b_n$ (at least for all but a finite number of $n$). If $\sum_{k=1}^{\infty} b_k$ converges then $\sum_{k=1}^{\infty} a_k$ also converges. If $\sum_{k=1}^{\infty} a_k$ diverges then $\sum_{k=1}^{\infty} b_k$ also diverges.

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Of course, the theorem *could* be re-written for series that are not necessarily non-negative using $|a_n| \leq |b_n|$ and substituting "converges absolutely" for "converges." (And with "diverges" replaced by "does not converge absolutely.")

**Theorem** Given two series $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ with $a_n \geq 0$, $b_n > 0$, and also

$$lim_{n\to\infty}\frac{a_n}{b_n} = L$$

where $0 < L < \infty$. Then $\sum_{k=1}^{\infty} b_k$ and $\sum_{k=1}^{\infty} a_k$ either both converge or both diverge.

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which is the same as

$$\frac{L}{2} b_n < a_n < 2L * b_n$$

**Ratio Test** For series $\sum_{k=1}^{\infty} a_k$ with all $a_k > 0$. If

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$$

exists and we set $L$ equal to this limit, then $L < 1$ implies that the series converges absolutely and $L > 1$ implies that the series diverges.

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$(1 + L)/2 < 1$ because...?

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$\frac{a_N}{r^N}$ is a positive constant, call it "$K$." All of this implies that

$$a_n < Kr^n$$

as long as $n \geq N$.

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Something similar would be done for $L > 1$, but with inequalities reversed to show divergence.

**Theorem (Root Test)** Given series $\sum_{k=1}^{\infty} a_k$ with $a_k \geq 0$. If

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The proof is in fact a bit simpler than for the ratio test!

**Theorem** Suppose that $f$ is a positive decreasing function and that $\sum_{k=1}^{\infty} a_k$ is a series for which $a_k = f(k)$. Then the series converges iff

$$\int_1^{\infty} f(x)dx < \infty.$$

Integral test, why is it true?

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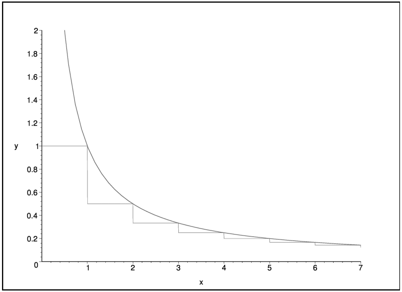

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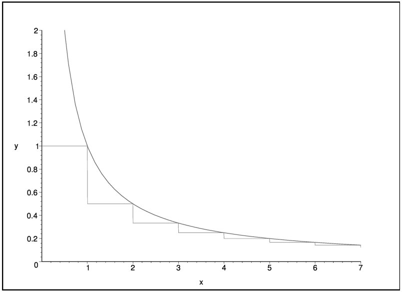


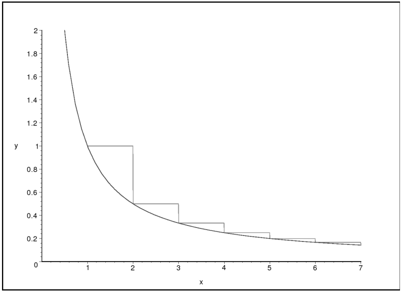

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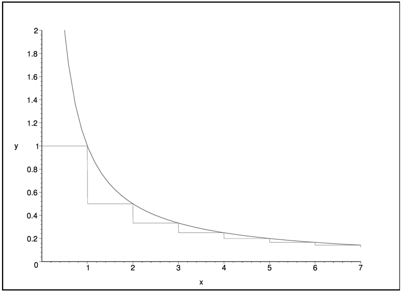


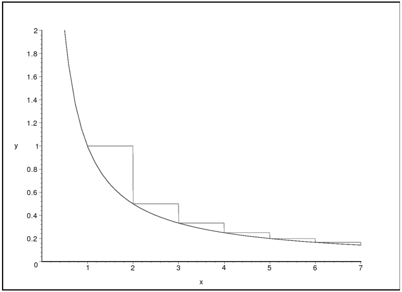


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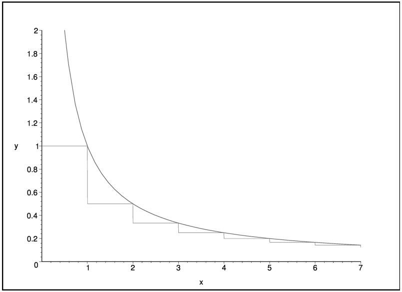


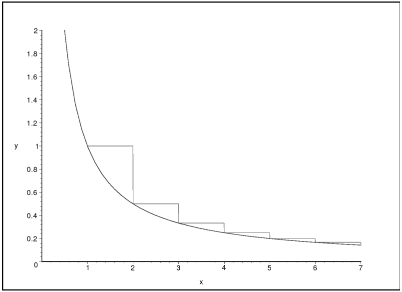


$$\sum_{k=1}^{n} a_k \leq a_1 + \int_1^n f(x)dx$$

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The integral test may be used to determine this. Alternatively, something called the *Cauchy Condensation Test* may be used. The Cauchy Condensation Test accomplishes more or less the same thing, but it is nice to be able to avoid talking about integrals.

**Theorem (Cauchy Condensation Test)** Suppose that

$$a_1 + a_2 + a_3 + \cdots$$

is a series whose summands are (eventually) positive and decreasing. This series converges if and only if the series

$$a_1 + 2a_2 + 4a_4 + 8a_8 + \cdots + 2^k a_{2^k} + \cdots$$

converges.

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The Integral Test is a bit more subtle and may be sometimes useful for series for which the Ratio and Root Tests fail. Notably, the so-called "p-series."