MATH 510, Notes 7

Modern Analysis

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An infinite series:

$$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + \cdots$$

A partial sum:

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n$$

$a_1, a_2, \cdots$ - *summands*

Saying that the series converges to T,

$$\sum_{k=1}^{\infty} a_k = T$$

means that the sequence of partial sums converges to T. (Given $\epsilon > 0$ there exists N so that ... etc.)

What do we know?

- $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges.
- $\sum_{k=0}^{\infty} r^k$ converges to $\frac{1}{1-r}$ if $|r| < 1$.
- $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$ converges.

Things we think we know:

If the summands are functions, sometimes it is OK to take the derivative or integral term by term, and sometimes not.

If the summands are continuous functions, sometimes the limit is continuous and sometimes not.

Sometimes mixing up the terms of a series changes the target/limit, and sometimes not.

When does a series converge?

Theorem If $\sum_{k=1}^{\infty} a_k$ converges, then $a_n \to 0$

But of course that only tells us when a series *might* converge.

Or, more often, that might tell us when a series does *not* converge.

Preview: We do not want to bypass the most straightforward sort of test about convergent or divergent series. The author does not mention until section 4.2, but let's put it out there now:

Theorem Given two infinite series $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$, suppose that for all values of k we have

$$0 \le a_k \le b_k.$$

If the series $\sum_{k=1}^{\infty} b_k$ converges, then $\sum_{k=1}^{\infty} a_k$ does also. If the series $\sum_{k=1}^{\infty} a_k$ diverges, then $\sum_{k=1}^{\infty} b_k$ does also.

Proof left to reader?

Can you see why theorem is still true if instead we know $0 \le a_k \le C \cdot b_k$ for some positive constant C? What if we only know that $0 \le a_k \le b_k$ "eventually?" That is, after some finite number of terms at the beginning of the series.

Definition An infinite sequence $\{S_1, S_2, ...\}$ is *Cauchy* if given $\epsilon > 0$ we can find a number N so that for any numbers n and m with $n > m \ge N$ we have

$$|S_n - S_m| < \epsilon$$

A series is *Cauchy* when its sequence of partial sums is Cauchy.

Theorem A sequence or series converges iff it is Cauchy.

To prove this, we need to think about the following ideas:
Given a *bounded* sequence $\{S_n\}$ let:
$U_1 = l.u.b\{S_1, S_2, S_3, \cdots\}$
$U_2 = l.u.b\{S_2, S_3, S_4, \cdots\}$
$U_3 = l.u.b\{S_3, S_4, S_5, \cdots\}$, etc.

In a similar way define a sequence $\{L_n\}$ using g.l.b. instead of l.u.b.

Facts:

- The $\{U_n\}$ and $\{L_n\}$ sequence are also bounded.
- The $\{U_n\}$ sequence is decreasing and the $\{L_n\}$ sequence is increasing.
- $U_n \ge L_n$ for all n.

As a bounded decreasing sequence, $\{U_n\}$ has a limit (it decreases to its greatest lower bound). This number is called the *limit superior* for the sequence $\{S_n\}$ and designated $\limsup_n S_n$.

Similarly there is a limit inferior $\liminf_n S_n$ defined using $\{L_n\}$

(Note: the idea of *limsup* and *liminf* may be extended to unbounded sequences if we allow *limsup* and *liminf* to take on the values $+\infty$ or $-\infty$.)

Theorem A sequence or series converges iff it is Cauchy.

The proof is not so bad. Converges $\Rightarrow$ Cauchy is easy. The other direction: Cauchy $\Rightarrow$ converges is more difficult.

One problem is that we do not know up front what should be the value for the target/limit. But we can get this value either using the nested interval principle applied to the intervals $[L_n, U_n]$, or more directly by showing that for a Cauchy sequence the limsup is the same as the liminf, and that this number is also the limit.

Definition Series $\sum_{k=1}^{\infty} a_k$ *converges absolutely* if $\sum_{k=1}^{\infty} |a_k|$ converges.

Is it obvious that if a series converges absolutely, then it converges in the usual sense?

If a series converges in the usual sense, does it converge absolutely?

If a series $\sum_{k=1}^{\infty} a_k$ converges to T, can we tell what $\sum_{k=1}^{\infty} |a_k|$ converges to?

Theorem If a series converges absolutely then the series converges in the usual sense. That is, if

$$\sum_{k=1}^{\infty} |a_k|$$

converges, then

$$\sum_{k=1}^{\infty} a_k$$

converges.

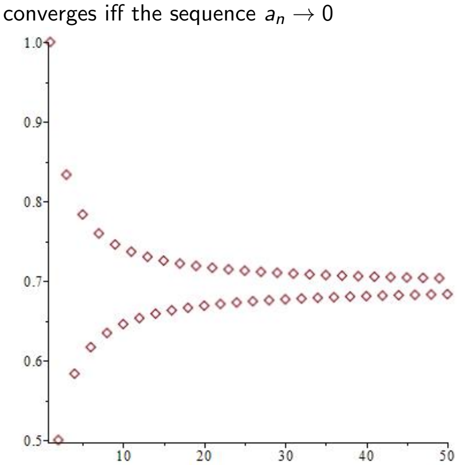
If a series converges but does not converge absolutely, then we say that the series *converges conditionally.*

An example of a series that converges conditionally?

Theorem If $a_1 \geq a_2 \geq a_3 \geq \cdots \geq 0$, then the alternating series

$$a_1 - a_2 + a_3 - a_4 + a_5 - \cdots$$

converges iff the sequence $a_n \to 0$



Just as little more perspective on absolute convergence:

For any real number a, define

$$a^+ = \begin{cases} a & : a \geq 0 \\ 0 & : a < 0 \end{cases}$$

and

$$a^- = \begin{cases} -a & : a < 0 \\ 0 & : a \geq 0 \end{cases}$$

Simple, one or the other of a^+ and a^- is zero, but note that:

$$a = a^+ - a^-$$

and

$$|a| = a^+ + a^-$$

It may be useful to think of an infinite series in the following way:

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} a_k^+ - \sum_{k=1}^{\infty} a_k^-$$

and

$$\sum_{k=1}^{\infty} |a_k| = \sum_{k=1}^{\infty} a_k^+ + \sum_{k=1}^{\infty} a_k^-$$

Similarly, the partial sums:

$$\sum_{k=1}^{n} a_k = \sum_{k=1}^{n} a_k^+ - \sum_{k=1}^{n} a_k^-$$

and

$$\sum_{k=1}^{n} |a_k| = \sum_{k=1}^{n} a_k^+ + \sum_{k=1}^{n} a_k^-$$

A series $\sum_{k=1}^{\infty} a_k$ converges absolutely IFF both $\sum_{k=1}^{\infty} a_k^+$ and $\sum_{k=1}^{\infty} a_k^-$ converge, in which case

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} a_k^+ - \sum_{k=1}^{\infty} a_k^-.$$

If a series converges conditionally, it must be the case that separately both $\sum_{k=1}^{\infty} a_k^+$ and $\sum_{k=1}^{\infty} a_k^-$ diverge, but somehow their interaction allows them to converge when the partial sums are subtracted.

Conditional convergence is a difficult thing to study in detail. The alternating series test is relatively easy to understand.But going beyond that requires some pretty sophisticated ideas.

So we will first concentrate on the absolute convergence, where a great deal can be done with so-called "comparison tests" and related tests that indirectly depend on comparisons.

In much of what follows, the theorems will assume that we have non-negative summands. That is, series $\sum_{k=1}^{\infty} a_k$ for which all $a_k \geq 0$.

But of course these theorems would apply to other not necessarily non-negative (is that a triple negative?!?) series if we are testing for absolute convergence by applying the theorems to $\sum_{k=1}^{\infty} |a_k|$.

Recall: **Theorem** Given two series $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ with $0 \leq a_n \leq b_n$ (at least for all but a finite number of n). If $\sum_{k=1}^{\infty} b_k$ converges then $\sum_{k=1}^{\infty} a_k$ also converges. If $\sum_{k=1}^{\infty} a_k$ diverges then $\sum_{k=1}^{\infty} b_k$ also diverges.

Of course, the theorem *could* be re-written for series that are not necessarily non-negative using $|a_n| \leq |b_n|$ and substituting "converges absolutely" for "converges."(And with "diverges" replaced by "does not converge absolutely.")

Theorem Given two series $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ with $a_n \geq 0$, $b_n > 0$, and also

$$lim_{n\to\infty} \frac{a_n}{b_n} = L$$

where $0 < L < \infty$. Then $\sum_{k=1}^{\infty} b_k$ and $\sum_{k=1}^{\infty} a_k$ either both converge or both diverge.

Why is this true?

The following idea could easily be turned into a formal proof: $\frac{a_n}{b_n} \approx L$ "eventually," that is, for large values of n. Meaning that for all but a finite number of n we can say that

$$\frac{1}{2}L < \frac{a_n}{b_n} < 2L$$

which is the same as

$$\frac{L}{2} b_n < a_n < 2L * b_n$$

Ratio Test For series $\sum_{k=1}^{\infty} a_k$ with all $a_k > 0$. If

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$$

exists and we set L equal to this limit, then $L < 1$ implies that the series converges absolutely and $L > 1$ implies that the series diverges.

Why is this true? Let $q_n = \frac{a_{n+1}}{a_n}$, and thus we know $q_n \to L$.

Suppose first that $L < 1$. Apply the definition of convergence with $\epsilon = (1 - L)/2$. Then we can say that there is a value of N so that for $n \geq N$ we have:

$$|q_n - L| < (1 - L)/2$$

and a little algebra shows that for these values of n

$$q_n < (1 + L)/2.$$

From this point on, keep in mind that N is a *constant*.

$(1 + L)/2 < 1$ because...?

Let $r = (1+L)/2$. We now know that for all sufficient large values of n we have $\frac{a_{n+1}}{a_n} < r$ or equivalently $\frac{a_n}{a_{n-1}} < r$ and hence $a_n < a_{n-1} * r$. So:

$$a_n < a_{n-1} * r < a_{n-2} * r^2 < \cdots < a_N * r^{n-N} = \frac{a_N}{r^N} r^n.$$

$\frac{a_N}{r^N}$ is a positive constant, call it "K." All of this implies that

$$a_n < Kr^n$$

as long as $n \geq N$. Is it clear what to do from here? (Comparison test, comparing with ...?)

Something similar would be done for $L > 1$, but with inequalities reversed to show divergence.

Theorem (Root Test) Given series $\sum_{k=1}^{\infty} a_k$ with $a_k \geq 0$. If

$$L = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$$

then $L < 1$ implies that the series converges absolutely and $L > 1$ implies that the series diverges.

The proof is in fact a bit simpler than for the ratio test!

Theorem Suppose that f is a positive decreasing function and that $\sum_{k=1}^{\infty} a_k$ is a series for which $a_k = f(k)$. Then the series converges iff

$$\int_1^{\infty} f(x)dx < \infty.$$

Integral test, why is it true?

$$\sum_{k=1}^{n} a_k \leq a_1 + \int_1^n f(x)dx$$

$$\int_1^n f(x)dx \leq \sum_{k=1}^{n-1} a_k$$

For what values of p does this series converge??

$$\sum_{k=1}^{\infty} \frac{1}{k^p}$$

The integral test may be used to determine this. Alternatively, something called the *Cauchy Condensation Test* may be used. The Cauchy Condensation Test accomplishes more or less the same thing, but it is nice to be able to avoid talking about integrals.

Theorem (Cauchy Condensation Test) Suppose that

$$a_1 + a_2 + a_3 + \cdots$$

is a series whose summands are (eventually) positive and decreasing. This series converges if and only if the series

$$a_1 + 2a_2 + 4a_4 + 8a_8 + \cdots + 2^k a_{2^k} + \cdots$$

converges.

Couple notes:

The Ratio Test and Root Test will only work with series that could theoretically be compared with geometric series.

The Integral Test is a bit more subtle and may be sometimes useful for series for which the Ratio and Root Tests fail. Notably, the so-called "p-series."