MATH 510, Notes 8

Modern Analysis

James Madison University

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It is pretty common to see $\sum_{k=1}^{\infty} f_k$ and $\sum_{k=1}^{\infty} f_k(x)$ used interchangeably. Technically, the latter should refer to the functions evaluated at a particular point x, but nonetheless ...

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- $\cos(\frac{\pi x}{2}) - \frac{1}{3}\cos(\frac{3\pi x}{2}) + \frac{1}{5}\cos(\frac{5\pi x}{2}) - \cdots = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{2k-1}\cos(\frac{(2k-1)\pi x}{2})$

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The second and third are examples of *power series*.

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It will turn out that a power series $\sum_{n=0}^{\infty} a_n x^n$ either converges everywhere or for x in some interval centered at 0 or possibly only at $x = 0$. When there is a finite interval, this interval turns out to be the values of x for which we can use a comparison test with a geometric series.

Theorem Given a power series $\sum_{n=0}^{\infty} a_n x^n$, suppose that

$$0 < \limsup_{n\to\infty} \sqrt[n]{|a_n|} < \infty.$$

Let

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If $\limsup_{n\to\infty} \sqrt[n]{|a_n|} = 0$ then the series converges absolutely for all x. If $\limsup_{n\to\infty} \sqrt[n]{|a_n|} = \infty$ then the series converges only for $x = 0$.

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Note: If $\lim_{n\to\infty} \frac{a_n}{a_{n+1}}$ exists, then $R = \lim_{n\to\infty} \frac{a_n}{a_{n+1}}$.

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$$\limsup_{n\to\infty} \sqrt[n]{|a_n * x^n|} = \limsup_{n\to\infty} \sqrt[n]{|a_n|} * |x|$$
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That is, $\frac{1}{R}|x| < 1$ I.e. $|x| < R$

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Generalize the alternating series test?

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Generalize the alternating series test?

The following is stated in reference to a series, but in reality it is strictly speaking an algebra exercise with finite sums:

Abel's Lemma Given a series of the form

$$\sum_{k=1}^{\infty} a_k b_k$$

where

$$b_1 \geq b_2 \geq b_3 \geq ... \geq 0.$$

Let

$$S_n = \sum_{k=1}^{n} a_k.$$

If there is a number M so that $|S_n| \leq M$ for all n (that is, the sequence S_n is **bounded**) then

$$|\sum_{k=1}^{n} a_k b_k| \leq M b_1$$

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$$\begin{aligned}
\sum_{k=1}^{n} a_k b_k &= S_1 b_1 + (S_2 - S_1) b_2 + \cdots + (S_n - S_{n-1}) b_n \\
&= (S_1 b_1 + S_2 b_2 + \cdots + S_n b_n) - (S_1 b_2 + S_2 b_3 + \cdots + S_{n-1} b_n) \\
&= S_1(b_1 - b_2) + S_2(b_2 - b_3) + \cdots + S_{n-1}(b_{n-1} - b_n) + S_n b_n \\
&= \sum_{k=1}^{n-1} S_k (b_k - b_{k+1}) + S_n b_n
\end{aligned}$$

$$
\begin{aligned}
|\sum_{k=1}^{n} a_k b_k| &\leq \sum_{k=1}^{n-1} |S_k(b_k - b_{k+1})| + |S_n b_n| \\
&= \sum_{k=1}^{n-1} |S_k|(b_k - b_{k+1}) + |S_n| b_n \\
&\leq \sum_{k=1}^{n-1} M(b_k - b_{k+1}) + M b_n \\
&= M(b_1 - b_2 + b_2 - b_3 + \cdots + b_{n-1} - b_n + b_n) \\
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Dirichlet's Test Given a series of the form

$$\sum_{k=1}^{\infty} a_k b_k$$

where

$$b_1 \geq b_2 \geq b_3 \geq ... \geq 0$$

and in addition $b_k \to 0$. Let

$$S_n = \sum_{k=1}^{n} a_k.$$

If there is a number M so that $|S_n| \leq M$ for all n (that is, the sequence S_n is **bounded**) then the original series $\sum_{k=1}^{\infty} a_k b_k$ **converges**.

You should think of this as a generalization of the alternating series test.

Indeed, this is exactly the alternating series test if we let $a_k = (-1)^{k+1}$, since then the partial sums $S_n = \sum_{k=1}^{n} a_k$ would be either ... ??

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The following would apply to any series, but for the specific type we are talking about now ... *Informally*, we can think of

$$\sum_{k=1}^{\infty} a_k b_k = \sum_{k=1}^{n} a_k b_k + \sum_{k=n+1}^{\infty} a_k b_k$$

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$$\sum_{k=1}^{\infty} a_k b_k = \sum_{k=1}^{n} a_k b_k + \sum_{k=n+1}^{\infty} a_k b_k$$

The series converges to a target value T if the sequence of partial sums $\sum_{k=1}^{n} a_k b_k$ converges to T, which is really the same as saying that the leftover tail $\sum_{k=n+1}^{\infty} a_k b_k \to 0$.

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The idea of a Cauchy sequence is merely a way to talk about this more carefully using finite sums instead of infinite sums.

$$\sum_{k=1}^{\infty} a_k b_k = \sum_{k=1}^{n} a_k b_k + \sum_{k=n+1}^{\infty} a_k b_k$$

What we really want to do is apply Abel's lemma not to the entire series, but to the tail. This tells us that for any larger number $m > n$ we have

$$|\sum_{k=n+1}^{m} a_k b_k| \leq b_{n+1} M^*$$

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(note: the number M^* may not be the same M as in the statement of the theorem. If it is not clear why, you can see why in reading the proof. But in fact, the number M^* need be no bigger than $2M$.)

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The right hand side above does not depend on m, so again thinking a little informally, we can imagine letting $m \to \infty$ giving us

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That is, the "tail" is less than or equal to $b_{n+1}M^*$. Since we know that $b_n \to 0$, this tells us that the "tail" goes to zero. A more formal proof would make us clean this up a little, but that is the idea.

Some identities:

$$\cos(y) - \cos(3y) + \cos(5y) - \cdots + (-1)^{n+1}\cos((2n-1)y) = \frac{1-(-1)^n\cos(2ny)}{2\cos(y)}$$

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$$\sin(y) + \sin(2y) + \sin(3y) + \sin(4y) + \cdots + \sin(ny) = \frac{\sin(y)}{2}\left(\frac{1-\cos(ny)}{1-\cos(y)}\right) + \frac{\sin(ny)}{2}$$

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$$\sin(\alpha) + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \cdots + \sin(\alpha+n\beta) = \frac{\sin(\frac{(n+1)\beta}{2})\sin(\alpha+\frac{n\beta}{2})}{\sin(\frac{n\beta}{2})}$$