MATH 510, Notes 8

Modern Analysis

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A sequence of functions S_n converges **pointwise** to a limit function S if *for each individual* x the sequence $S_n(x)$ converges to $S(x)$.

The "point" of "pointwise convergence" is that the rate at which $S_n(x) \to S(x)$ may be quite different at different values of x.

When we say that a *series* of functions

$$\sum_{k=1}^{\infty} f_k = f_1 + f_2 + f_3 + \cdots$$

converges (pointwise) we mean that its sequence of partial sums converges (pointwise).

It is pretty common to see $\sum_{k=1}^{\infty} f_k$ and $\sum_{k=1}^{\infty} f_k(x)$ used interchangeably. Technically, the latter should refer to the functions evaluated at a particular point x, but nonetheless ...

Examples:

- 3.3.25 $S_n(x) = \frac{\ln(x+2) - x^{2n} * \sin(x)}{1+x^{2n}}$
- Geometric series $1 + x + x^2 + x^3 + \cdots = \sum_{k=0}^{\infty} x^k$
- $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!}$
- $\cos(\frac{\pi x}{2}) - \frac{1}{3}\cos(\frac{3\pi x}{2}) + \frac{1}{5}\cos(\frac{5\pi x}{2}) - \cdots = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{2k-1}\cos(\frac{(2k-1)\pi x}{2})$

The second and third are examples of *power series*.

Power series:

$$\sum_{n=0}^{\infty} a_n x^n$$

or more generally

$$\sum_{n=0}^{\infty} a_n (x-c)^n,$$

but for simplicity we concentrate on $\sum_{n=0}^{\infty} a_n x^n$.

It will turn out that a power series $\sum_{n=0}^{\infty} a_n x^n$ either converges everywhere or for x in some interval centered at 0 or possibly only at $x = 0$. When there is a finite interval, this interval turns out to be the values of x for which we can use a comparison test with a geometric series.

Theorem Given a power series $\sum_{n=0}^{\infty} a_n x^n$, suppose that

$$0 < \limsup_{n\to\infty} \sqrt[n]{|a_n|} < \infty.$$

Let

$$R = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|a_n|}}.$$

Then the series converges absolutely for $|x| < R$ and diverges for $|x| > R$.

If $\limsup_{n\to\infty} \sqrt[n]{|a_n|} = 0$ then the series converges absolutely for all x. If $\limsup_{n\to\infty} \sqrt[n]{|a_n|} = \infty$ then the series converges only for $x = 0$.
Note: If $\lim_{n\to\infty} \frac{a_n}{a_{n+1}}$ exists, then $R = \lim_{n\to\infty} \frac{a_n}{a_{n+1}}$.

Why is this true? For

$$\sum_{n=0}^{\infty} a_n x^n$$

apply the root test.

$$\limsup_{n\to\infty} \sqrt[n]{|a_n * x^n|} = \limsup_{n\to\infty} \sqrt[n]{|a_n|} * |x|$$
$$= \frac{1}{R}|x|$$

The series converges if the limit superior in the root test is less than ...?

That is, $\frac{1}{R}|x| < 1$ I.e. $|x| < R$

Back to convergence when comparisons are not realistic...

Generalize the alternating series test?

The following is stated in reference to a series, but in reality it is strictly speaking an algebra exercise with finite sums:

Abel's Lemma Given a series of the form

$$\sum_{k=1}^{\infty} a_k b_k$$

where

$$b_1 \geq b_2 \geq b_3 \geq \ldots \geq 0.$$

Let

$$S_n = \sum_{k=1}^{n} a_k.$$

If there is a number M so that $|S_n| \leq M$ for all n (that is, the sequence S_n is **bounded**) then

$$|\sum_{k=1}^{n} a_k b_k| \leq M b_1$$

Why is this true?

This is merely a clever rewriting and rearrangement of the sum. You could work it out from scratch if you had enough time to contemplate it. Might take a while, but still... Basic fact is that $a_k = S_k - S_{k-1}$: (And it makes some sense to set $S_0 = 0$.)

$$\begin{aligned}\sum_{k=1}^{n} a_k b_k &= S_1 b_1 + (S_2 - S_1) b_2 + \cdots + (S_n - S_{n-1}) b_n \\ &= (S_1 b_1 + S_2 b_2 + \cdots + S_n b_n) - (S_1 b_2 + S_2 b_3 + \cdots + S_{n-1} b_n) \\ &= S_1 (b_1 - b_2) + S_2 (b_2 - b_3) + \cdots + S_{n-1}(b_{n-1} - b_n) + S_n b_n \\ &= \sum_{k=1}^{n-1} S_k (b_k - b_{k+1}) + S_n b_n\end{aligned}$$

$$\begin{aligned}|\sum_{k=1}^{n} a_k b_k| &\leq \sum_{k=1}^{n-1} |S_k (b_k - b_{k+1})| + |S_n b_n| \\ &= \sum_{k=1}^{n-1} |S_k| (b_k - b_{k+1}) + |S_n| b_n \\ &\leq \sum_{k=1}^{n-1} M (b_k - b_{k+1}) + M b_n \\ &= M(b_1 - b_2 + b_2 - b_3 + \cdots + b_{n-1} - b_n + b_n) \\ &= M b_1\end{aligned}$$

Dirichlet's Test Given a series of the form

$$\sum_{k=1}^{\infty} a_k b_k$$

where

$$b_1 \geq b_2 \geq b_3 \geq \ldots \geq 0$$

and in addition $b_k \to 0$. Let

$$S_n = \sum_{k=1}^{n} a_k.$$

If there is a number M so that $|S_n| \leq M$ for all n (that is, the sequence S_n is **bounded**) then the original series $\sum_{k=1}^{\infty} a_k b_k$ **converges**.

You should think of this as a generalization of the alternating series test.

Indeed, this is exactly the alternating series test if we let $a_k = (-1)^{k+1}$, since then the partial sums $S_n = \sum_{k=1}^{n} a_k$ would be either ... ??

Why is this true?

With all series, the question of whether the series converges is really the same as asking does the "tail" of the series "get sufficiently small".

The following would apply to any series, but for the specific type we are talking about now ... *Informally*, we can think of

$$\sum_{k=1}^{\infty} a_k b_k = \sum_{k=1}^{n} a_k b_k + \sum_{k=n+1}^{\infty} a_k b_k$$

The series converges to a target value T if the sequence of partial sums $\sum_{k=1}^{n} a_k b_k$ converges to T, which is really the same as saying that the leftover tail $\sum_{k=n+1}^{\infty} a_k b_k \to 0$.

The idea of a Cauchy sequence is merely a way to talk about this more carefully using finite sums instead of infinite sums.

$$\sum_{k=1}^{\infty} a_k b_k = \sum_{k=1}^{n} a_k b_k + \sum_{k=n+1}^{\infty} a_k b_k$$

What we really want to do is apply Abel's lemma not to the entire series, but to the tail. This tells us that for any larger number $m > n$ we have

$$|\sum_{k=n+1}^{m} a_k b_k| \leq b_{n+1} M^*$$

(note: the number M^* may not be the same M as in the statement of the theorem. If it is not clear why, you can see why in reading the proof. But in fact, the number M^* need be no bigger than $2M$.)

$$|\sum_{k=n+1}^{m} a_k b_k| \leq b_{n+1} M^*$$

The right hand side above does not depend on m, so again thinking a little informally, we can imagine letting $m \to \infty$ giving us

$$|\sum_{k=n+1}^{\infty} a_k b_k| \leq b_{n+1} M^*.$$

That is, the "tail" is less than or equal to $b_{n+1} M^*$. Since we know that $b_n \to 0$, this tells us that the "tail" goes to zero. A more formal proof would make us clean this up a little, but that is the idea.

Some identities:

$$\cos(y) - \cos(3y) + \cos(5y) - \cdots + (-1)^{n+1} \cos((2n-1)y) = \frac{1 - (-1)^n \cos(2ny)}{2\cos(y)}$$

$$\sin(y) + \sin(2y) + \sin(3y) + \sin(4y) + \cdots + \sin(ny) = \frac{\sin(y)}{2}\left(\frac{1-\cos(ny)}{1-\cos(y)}\right) + \frac{\sin(ny)}{2}$$

$$\sin(\alpha) + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \cdots + \sin(\alpha+n\beta) = \frac{\sin(\frac{(n+1)\beta}{2}) \sin(\alpha + \frac{n\beta}{2})}{\sin(\frac{n\beta}{2})}$$