MATH 510, Notes 9

Modern Analysis

James Madison University

A *regrouping* of an infinite series such as

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would result from some arbitrary insertion of parentheses, for example:

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$$b_1 = (a_1 + a_2)$$

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A related question: If a series diverges, does every regrouping of that series diverge?

Theorem If $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ both converge, then so does $\sum_{k=1}^{\infty}(a_k + b_k)$, and

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Another result, for the record:
Theorem If $\sum_{k=1}^{\infty} a_k$ converges and c is any number, then $\sum_{k=1}^{\infty} c \cdot a_k$ also converges and

$$\sum_{k=1}^{\infty} c \cdot a_k = c \sum_{k=1}^{\infty} a_k.$$

A *rearrangement* of an infinite series is a bit more confusing, resulting from taking all of the same terms from the original series in forming a new series, but in a different order. For example, for the series:

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There is not necessarily a pattern. For $\sum_{k=1}^{\infty} b_k$ to be a rearrangment of $\sum_{k=1}^{\infty} a_k$, all we need is a one-to-one and onto function $\gamma : \mathbb{N} \to \mathbb{N}$ so that $b_n = a_{\gamma(n)}$.

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All we would really need to do is find the largest of $\gamma(1), \gamma(2), \cdots, \gamma(1000)$ and be sure that we include at least that many terms in the partial sum.

One can deal with not necessarily positive series using the Cauchy series ideas, but perhaps it is simpler to remember that we can break a series into positive and negative parts:

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And recall that $\sum_{k=1}^{\infty} a_k$ converges absolutely *iff* the two positive series on the right side converge.

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and that the series converges to $G(x)$ given by

$$G(x) = \begin{cases} 1 & : -1 < x < 1 \\ 0 & : x = \pm 1 \\ -1 & : x < -1 \text{ or } x > 1 \end{cases}$$

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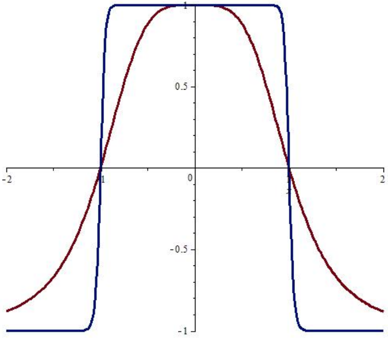

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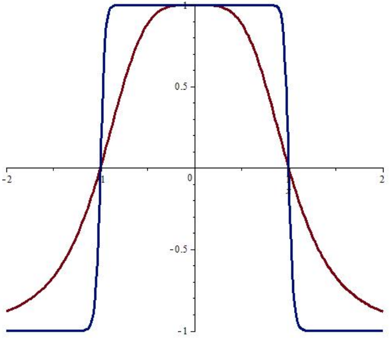

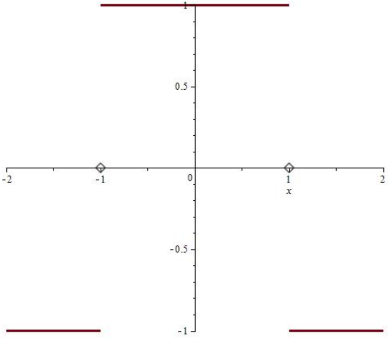

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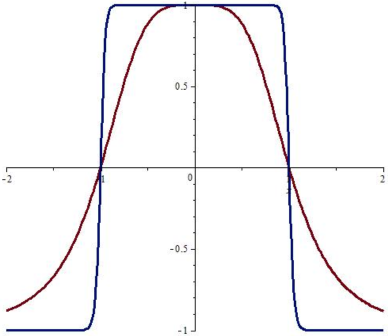

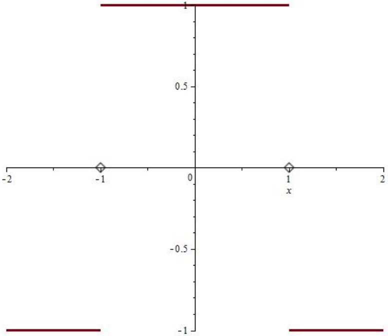

For series whose partial sums are continuous, apparently the limit need not be continuous(?)

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Since $S_n(x) \to G(x)$, we can find a number N so that $|S_n(x) - G(x)| < \frac{\epsilon}{3}$ when $n \geq N$ (and in particular when $n = N$). Since S_N is continuous, we can find δ so that $|x - a| < \delta$ implies $|S_N(x) - S_N(a)| < \frac{\epsilon}{3}$. Thus if $|x - a| < \delta$ then:

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This would imply that function G is continuous, but based on our example this cannot be???

Definition For a series of functions

$$\sum_{k=1}^{\infty} g_k(x)$$

with partial sums

$$S_n(x) = \sum_{k=1}^{n} g_k(x)$$

we say that the series *converges uniformly* to function G on set E if given $\epsilon > 0$ we can find a number N so that $|S_n(x) - G(x)| < \epsilon$ **for all** $x \in E$ when $n \geq N$.

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The key here is that for any ϵ we can find a single N that "works" for all $x \in E$. This is not to say that the series somehow converges at the same "rate" for all x; more often we can think of this as meaning that there is a "worst case" and that once we identify that worst case then convergence is at least that fast at every value of x.

Theorem For a series of functions $\sum_{k=1}^{\infty} g_k(x)$, if each $g_k(x)$ is continuous for x in some interval I and the series converges uniformly on I to function G, then G is also continuous.

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What about derivatives and integrals?

There are many examples of series

$$\sum_{k=1}^{\infty} g_k(x)$$

that converge (pointwise), but

$$\int_a^b \sum_{k=1}^{\infty} g_k(x)dx \neq \sum_{k=1}^{\infty} \int_a^b g_k(x)dx$$

Nonetheless:

Theorem For a series of functions $\sum_{k=1}^{\infty} g_k(x)$, suppose that each $g_k(x)$ is integrable on an interval $[a, b]$ and the series converges uniformly on $[a, b]$ to function G, then G is also integrable and

$$\int_a^b G(x)dx = \sum_{k=1}^{\infty} \int_a^b g_k(x)dx.$$

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That is:

$$\int_a^b \sum_{k=1}^{\infty} g_k(x)dx = \sum_{k=1}^{\infty} \int_a^b g_k(x)dx$$

The same theorem may be stated in a "sequence" version:

For a sequence of functions $S_n(x)$, suppose that each S_n is integrable on an interval $[a, b]$ and the sequence converges uniformly on $[a, b]$ to function S. Then S is also integrable and

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Why is this true?

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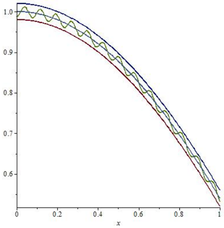
The proof is not difficult to write out, but at a basic level we can think of this graphically.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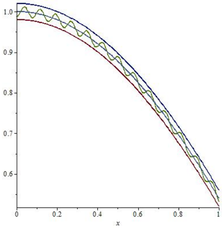
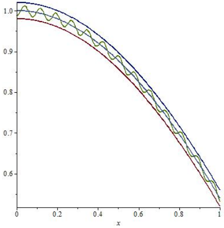


In the graph, imagine that the smooth curve is the limit function accompanied by an ϵ-band. With uniform convergence, partial sums (the squiggly curve) eventually must be inside the band. Thinking of areas related to integrals, functions with graphs inside the band will have integral very close together.



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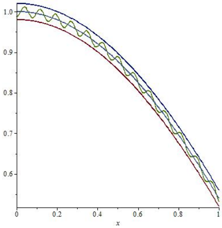
The basic inequality: if $|G(x) - \sum_{k=1}^{n} g_k(x)| < \frac{\epsilon}{b-a}$ then



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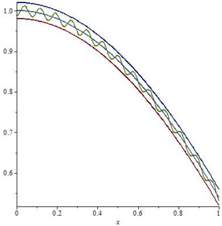

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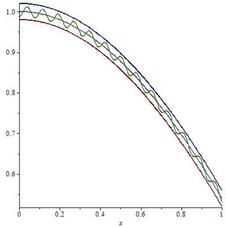

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If the limit function has a derivative, it may be that the derivative of the limit is not the limit of the derivatives.

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Also $g_k(x) = \int_a^x g_k'(t)dt + g_k(a)$ (why?)

From here, the primary thing would be to show why

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There are several more general versions of the following, but here is the most useful form:

Theorem (Weierstrauss M-test For a series of functions $\sum_{k=1}^{\infty} g_k(x)$, suppose that we can find constants $M_1, M_2, \cdots$ so that for all x in some interval I and for all k we have

$$|g_k(x)| \leq M_k.$$

If $\sum_{k=1}^{\infty} M_k$ converges then $\sum_{k=1}^{\infty} g_k(x)$ converges uniformly.

A consequence of the Weierstrauss M-test:

Theorem Given a power series

$$\sum_{k=0}^{\infty} a_k x^k = a_0 + a_1 x + a_2 x^2 + \cdots$$

with radius of convergence $R > 0$. Suppose that α is a number with $0 < \alpha < R$. Then the power series converges uniformly on $[-\alpha, \alpha]$.

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$$(\sum_{k=1}^{\infty} a_k x^k)' = \sum_{k=1}^{\infty} k a_{k-1} x^{k-1} = a_1 + 2a_2 x + 3a_3 x^2 + \cdots$$

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$$(\sum_{k=1}^{\infty} a_k x^k)' = \sum_{k=1}^{\infty} k a_{k-1} x^{k-1} = a_1 + 2a_2 x + 3a_3 x^2 + \cdots$$

Also, when $|x| < R$ we have:

$$\int_0^x \sum_{k=0}^{\infty} a_k t^k dt = a_0 x + \frac{a_1}{2} x^2 + \frac{a_2}{3} x^3 + \cdots$$

Dirichlet's Test Given a series of the form

$$\sum_{k=1}^{\infty} g_k(x)h_k(x)$$

where g_k and h_k are functions defined on some interval I. Suppose that

$$h_1(x) \geq h_2(x) \geq h_3(x) \geq \ldots \geq 0$$

and that there is a sequence of numbers $B_1 \geq B_2 \geq B_3 \geq \ldots$ with $B_k \geq h_k(x)$ for all $x \in I$, and $B_k \to 0$.
Let

$$S_n(x) = \sum_{k=1}^{n} g_k(x).$$

If there is a number M so that $|S_n(x)| \leq M$ for all n and all $x \in I$ then the original series $\sum_{k=1}^{\infty} g_k(x)h_k(x)$ **converges uniformly**.

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Theorem Suppose that a series of functions $\sum_{k=1}^{\infty} g_k(x)$ converges uniformly on the interval (a, b) and that each $g_k(x)$ is continuous on $[a, b]$. Then $\sum_{k=1}^{\infty} g_k(x)$ converges (uniformly) on the entire closed interval $[a, b]$.

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Theorem Suppose that a series of functions $\sum_{k=1}^{\infty} g_k(x)$ converges uniformly on the interval (a, b) and that each $g_k(x)$ is continuous on $[a, b]$. Then $\sum_{k=1}^{\infty} g_k(x)$ converges (uniformly) on the entire closed interval $[a, b]$.

Theorem Given a power series

$$\sum_{k=0}^{\infty} a_k x^k = a_0 + a_1 x + a_2 x^2 + \cdots$$

with radius of convergence $R > 0$. If the series converges at $x = R$, then it converges uniformly on $[0, R]$. If the series converges at $x = -R$, then it converges uniformly on $[-R, 0]$.