MATH 510, Notes 9

Modern Analysis

James Madison University

A *regrouping* of an infinite series such as

$$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + a_9 + a_{10} + \cdots$$

would result from some arbitrary insertion of parentheses, for example:

$$(a_1 + a_2) + (a_3 + a_4 + a_5 + a_6) + (a_7 + a_8) + a_9 + (a_{10} + a_{11}) + \cdots$$

essentially creating a new related series $\sum_{k=1}^{\infty} b_k$ where in this case:

$$b_1 = (a_1 + a_2)$$

$$b_2 = (a_3 + a_4 + a_5 + a_6)$$

$$b_3 = (a_7 + a_8)$$

etc.

Theorem If a series converges, every regrouping of that series converges to the same target value.

A proof would necessarily go back to the $\epsilon - N$ definition, but more than anything else it would be a question of a getting an appropriate notation to describe the choice of N.

A related question: If a series diverges, does every regrouping of that series diverge?

Theorem If $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ both converge, then so does $\sum_{k=1}^{\infty}(a_k + b_k)$, and

$$\sum_{k=1}^{\infty}(a_k + b_k) = \sum_{k=1}^{\infty} a_k + \sum_{k=1}^{\infty} b_k.$$

Is the converse of this theorem true? (What is the converse of this theorem?)

Another result, for the record:
Theorem If $\sum_{k=1}^{\infty} a_k$ converges and c is any number, then $\sum_{k=1}^{\infty} c \cdot a_k$ also converges and

$$\sum_{k=1}^{\infty} c \cdot a_k = c \sum_{k=1}^{\infty} a_k.$$

A *rearrangement* of an infinite series is a bit more confusing, resulting from taking all of the same terms from the original series in forming a new series, but in a different order. For example, for the series:

$$a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + a_9 + \cdots$$

The series

$$a_1 + a_3 + a_2 + a_5 + a_7 + a_4 + a_9 + a_{11} + a_6 + \cdots$$

would be a rearrangement.

There is not necessarily a pattern. For $\sum_{k=1}^{\infty} b_k$ to be a rearrangment of $\sum_{k=1}^{\infty} a_k$, all we need is a one-to-one and onto function $\gamma : \mathbb{N} \to \mathbb{N}$ so that $b_n = a_{\gamma(n)}$.

Theorem If a series converges absolutely, every rearrangement of that series converges absolutely to the same target value.

Why is this true?

For series with all positive terms, it is not difficult to make sense of the theorem.For a series $\sum_{k=1}^{\infty} a_k$ with all positive terms, the partial sums are always increasing and getting progressively closer to the target value. So, for example, if 1000 terms from the original series are required to get within a certain error tolerance of the target value, we could be sure that the rearranged series is well within that same error tolerance if we take a sufficient number of the "rearranged" terms so that at least the 1000 terms from the original series are included, albeit in a different order.

All we would really need to do is find the largest of $\gamma(1), \gamma(2), \cdots, \gamma(1000)$ and be sure that we include at least that many terms in the partial sum.

One can deal with not necessarily positive series using the Cauchy series ideas, but perhaps it is simpler to remember that we can break a series into positive and negative parts:

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} a_k^+ - \sum_{k=1}^{\infty} a_k^-$$

And recall that $\sum_{k=1}^{\infty} a_k$ converges absolutely *iff* the two positive series on the right side converge.

Theorem If a series converges conditionally, then for any real number T there is some rearrangement of that series that converges to T.

At first glance, that seems pretty odd. For example, for the alternating harmonic series

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$$

we are pretty sure that the target value is $\ln(2)$.

This theorem say that it is possible to find a rearrangement of $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$ that converge to 1, another rearrangement that converges to -10, another rearrangement that converges to 10000, and so on (and on!).

Why is this true?

Writing out a formal proof is a headache, but in fact the idea is pretty simple: For a *conditionally convergent* series, when we break up the **partial sums**

$$\sum_{k=1}^{n} a_k = \sum_{k=1}^{n} a_k^+ - \sum_{k=1}^{n} a_k^-$$

then we know that the two positive series related to the two sums on the right side both must diverge (to infinity).

$$\sum_{k=1}^{n} a_k = \sum_{k=1}^{n} a_k^+ - \sum_{k=1}^{n} a_k^-$$

For any possible target T, to begin we merely need to take enough positive terms until we get a sum greater than T. (Of course, if $T \leq 0$ that mean that the required number of positive terms is zero!) We are sure that we can do this, since we know that the partial sums of positive terms eventually run off to infinity. Next, throw in enough negative terms so the the partial sums drop below T. After that, more positive terms until we are above T, and so on.

Consider the series:

$$\sum_{k=1}^{\infty} \frac{2x^{2(k-1)}(1-x^2)}{(1+x^{2k})(1+x^{2(k-1)})}$$

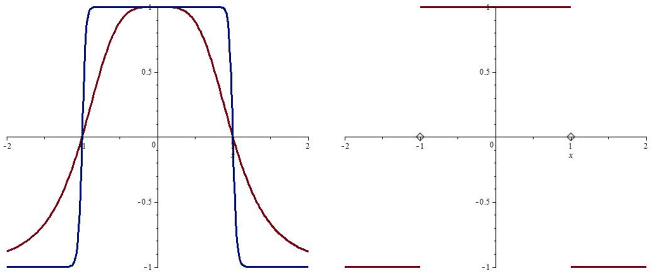
We have shown that the partial sums simplify as follows:

$$\sum_{k=1}^{n} \frac{2x^{2(k-1)}(1-x^2)}{(1+x^{2k})(1+x^{2(k-1)})} = \frac{1-x^{2n}}{1+x^{2n}}$$

and that the series converges to $G(x)$ given by

$$G(x) = \begin{cases} 1 & : -1 < x < 1 \\ 0 & : x = \pm 1 \\ -1 & : x < -1 \text{ or } x > 1 \end{cases}$$

A graph of partial sums with $n = 2$ and $n = 20$, and the series limit $G(x)$:



For series whose partial sums are continuous, apparently the limit need not be continuous(?)

But, what is wrong with the following "proof?" Let $S_n(x) = \frac{1-x^{2n}}{1+x^{2n}}$ be the partial sums, and recall that $G(x)$ is the series limit:
To attempt to "show" that G is continuous at point a, given $\epsilon > 0$ we would need to find a corresponding $\delta > 0$ so that $|x - a| < \delta$ would imply $|G(x) - G(a)| < \epsilon$.

Since $S_n(x) \to G(x)$, we can find a number N so that $|S_n(x) - G(x)| < \frac{\epsilon}{3}$ when $n \geq N$ (and in particular when $n = N$). Since S_N is continuous, we can find δ so that $|x - a| < \delta$ implies $|S_N(x) - S_N(a)| < \frac{\epsilon}{3}$. Thus if $|x - a| < \delta$ then:

$$\begin{aligned} |G(x) - G(a)| &= |G(x) - S_N(x) + S_N(x) - S_N(a) + S_N(a) - G(a)| \\ &\leq |G(x) - S_N(x)| + |S_N(x) - S_N(a)| + |S_N(a) - G(a)| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \end{aligned}$$

This would imply that function G is continuous, but based on our example this cannot be???

Definition For a series of functions

$$\sum_{k=1}^{\infty} g_k(x)$$

with partial sums

$$S_n(x) = \sum_{k=1}^{n} g_k(x)$$

we say that the series *converges uniformly* to function G on set E if given $\epsilon > 0$ we can find a number N so that $|S_n(x) - G(x)| < \epsilon$ **for all** $x \in E$ when $n \geq N$.

The key here is that for any ϵ we can find a single N that "works" for all $x \in E$. This is not to say that the series somehow converges at the same "rate" for all x; more often we can think of this as meaning that there is a "worst case" and that once we identify that worst case then convergence is at least that fast at every value of x.

Theorem For a series of functions $\sum_{k=1}^{\infty} g_k(x)$, if each $g_k(x)$ is continuous for x in some interval I and the series converges uniformly on I to function G, then G is also continuous.

Let $S_n(x) = \sum_{k=1}^{n} g_k(x)$. Since $S_n(x) \to G(x)$, we can find a number N so that $|S_n(x) - G(x)| < \frac{\epsilon}{3}$ **for all** $x \in I$ when $n \geq N$ (and in particular when $n = N$). Since S_N is continuous, we can find δ so that $|x - a| < \delta$ implies $|S_N(x) - S_N(a)| < \frac{\epsilon}{3}$. Thus if $|x - a| < \delta$ then:

$$\begin{aligned} |G(x) - G(a)| &= |G(x) - S_N(x) + S_N(x) - S_N(a) + S_N(a) - G(a)| \\ &\leq |G(x) - S_N(x)| + |S_N(x) - S_N(a)| + |S_N(a) - G(a)| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \end{aligned}$$

So...with pointwise convergence of continuous functions, the limit may or may not be continuous.

With uniform convergence of continuous functions, the limit *will* be continuous.

What about derivatives and integrals?

There are many examples of series

$$\sum_{k=1}^{\infty} g_k(x)$$

that converge (pointwise), but

$$\int_a^b \sum_{k=1}^{\infty} g_k(x) dx \neq \sum_{k=1}^{\infty} \int_a^b g_k(x) dx$$

Nonetheless:

Theorem For a series of functions $\sum_{k=1}^{\infty} g_k(x)$, suppose that each $g_k(x)$ is integrable on an interval $[a, b]$ and the series converges uniformly on $[a, b]$ to function G, then G is also integrable and

$$\int_a^b G(x)dx = \sum_{k=1}^{\infty} \int_a^b g_k(x)dx.$$

That is:

$$\int_a^b \sum_{k=1}^{\infty} g_k(x)dx = \sum_{k=1}^{\infty} \int_a^b g_k(x)dx$$

The same theorem may be stated in a "sequence" version:

For a sequence of functions $S_n(x)$, suppose that each S_n is integrable on an interval $[a, b]$ and the sequence converges uniformly on $[a, b]$ to function S. Then S is also integrable and

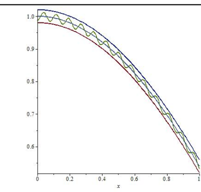
$$\int_a^b S(x)dx = \lim_{n\to\infty} \int_a^b S_n(x)dx$$

That is

$$\int_a^b \lim_{n\to\infty} S_n(x)dx = \lim_{n\to\infty} \int_a^b S_n(x)dx$$

Why is this true?

The proof is not difficult to write out, but at a basic level we can think of this graphically.



In the graph, imagine that the smooth curve is the limit function accompanied by an ϵ-band. With uniform convergence, partial sums (the squiggly curve) eventually must be inside the band. Thinking of areas related to integrals, functions with graphs inside the band will have integral very close together.
The basic inequality: if $|G(x) - \sum_{k=1}^{n} g_k(x)| < \frac{\epsilon}{b-a}$ then

$$\begin{aligned} \left|\int_a^b G(x)dx - \sum_{k=1}^{n} \int_a^b g_k(x)dx\right| &\leq \int_a^b |G(x)dx - \sum_{k=1}^{n} g_k(x)|dx \\ &< \int_a^b \frac{\epsilon}{b-a}dx = \epsilon \end{aligned}$$

For derivatives, the situations is, how to say, "more complicated."

Even with uniform limits, it may be that the derivative does not even exist for the limit function.

If the limit function has a derivative, it may be that the derivative of the limit is not the limit of the derivatives.

But uniform convergence does play a role:

Theorem For a series of functions $\sum_{k=1}^{\infty} g_k(x)$, suppose that we know that the series converges for at least one value of x (say $x = a$). In addition, suppose that each $g_k(x)$ is differentiable and the series *of derivatives* $\sum_{k=1}^{\infty} g_k'(x)$ converges uniformly on open interval I (containing point a). Then:

1. $\sum_{k=1}^{\infty} g_k(x)$ converges uniformly to a function G on the interval I.
2. G is differentiable.
3. $G'(x) = \sum_{k=1}^{\infty} g_k'(x)$

Why is this true?

Our text author gives an alternate proof, but you can informally think of this as related to the Fundamental Theorem of Calculus.Basic idea? Start by defining G as follows:

$$G(x) = \int_a^x \sum_{k=1}^{\infty} g_k'(t)dt + \sum_{k=1}^{\infty} g(a)$$

We know that for the series on the right

$$\int_a^x \sum_{k=1}^{\infty} g_k'(t)dt = \sum_{k=1}^{\infty} \int_a^x g_k'(t)dt$$

(why?)
Also $g_k(x) = \int_a^x g_k'(t)dt + g_k(a)$ (why?)

From here, the primary thing would be to show why

$$G(x) = \sum_{k=1}^{\infty} g_k(x)$$

A question we have (sort of) been ignoring: How to show that a series converges uniformly?

In examples so far, the partial sums of series we were looking at were either simple geometric series or the partial sum "miraculously" simplified into something we could figure out in an ad-hoc sort of way.

There are several more general versions of the following, but here is the most useful form:

Theorem (Weierstrauss M-test For a series of functions $\sum_{k=1}^{\infty} g_k(x)$, suppose that we can find constants $M_1, M_2, \cdots$ so that for all x in some interval I and for all k we have

$$|g_k(x)| \leq M_k.$$

If $\sum_{k=1}^{\infty} M_k$ converges then $\sum_{k=1}^{\infty} g_k(x)$ converges uniformly.

A consequence of the Weierstrauss M-test:
Theorem Given a power series

$$\sum_{k=0}^{\infty} a_k x^k = a_0 + a_1 x + a_2 x^2 + \cdots$$

with radius of convergence $R > 0$. Suppose that α is a number with $0 < \alpha < R$. Then the power series converges uniformly on $[-\alpha, \alpha]$. Furthermore, the function defined by this series is differentiable at every x with $|x| < R$ and

$$(\sum_{k=1}^{\infty} a_k x^k)' = \sum_{k=1}^{\infty} k a_{k-1} x^{k-1} = a_1 + 2a_2 x + 3a_3 x^2 + \cdots$$

Also, when $|x| < R$ we have:

$$\int_0^x \sum_{k=0}^{\infty} a_k t^k dt = a_0 x + \frac{a_1}{2} x^2 + \frac{a_2}{3} x^3 + \cdots$$

Dirichlet's Test Given a series of the form

$$\sum_{k=1}^{\infty} g_k(x) h_k(x)$$

where g_k and h_k are functions defined on some interval I. Suppose that

$$h_1(x) \geq h_2(x) \geq h_3(x) \geq ... \geq 0$$

and that there is a sequence of numbers $B_1 \geq B_2 \geq B_3 \geq ...$ with $B_k \geq h_k(x)$ for all $x \in I$, and $B_k \to 0$.
Let

$$S_n(x) = \sum_{k=1}^{n} g_k(x).$$

If there is a number M so that $|S_n(x)| \leq M$ for all n and all $x \in I$ then the original series $\sum_{k=1}^{\infty} g_k(x) h_k(x)$ **converges uniformly**.

Two more:
Theorem Suppose that a series of functions $\sum_{k=1}^{\infty} g_k(x)$ converges uniformly on the interval (a, b) and that each $g_k(x)$ is continuous on $[a, b]$. Then $\sum_{k=1}^{\infty} g_k(x)$ converges (uniformly) on the entire closed interval $[a, b]$.

Theorem Given a power series

$$\sum_{k=0}^{\infty} a_k x^k = a_0 + a_1 x + a_2 x^2 + \cdots$$

with radius of convergence $R > 0$. If the series converges at $x = R$, then it converges uniformly on $[0, R]$. If the series converges at $x = -R$, then it converges uniformly on $[-R, 0]$.