MATH 510, Fourier Series and Dirichlet's Theorem

Modern Analysis

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So … we are back to thinking about series similar to what was described back in Chapter 1, recalling the work of Fourier. We have theorems that, along with the appropriate algebra and trigonometric identities, can be used to demonstrate that these series converge. But it still remains an open question regarding whether the limit of those series is what we want it to be.

Here is what we are hoping for: Given some function f, we can find appropriate numbers a_n and b_n somehow related to f, so that the following series of functions converges:

$$F(x) = a_0 + \sum_{k=1}^{\infty} a_n \cos(kx) + b_n \sin(kx)$$

and that $f(x) = F(x)$, at least for x in some interval in which we are interested.

The text plays a little fast and loose with the distinction between the function we begin with, f, and the function defined by the series, F. We will try to maintain that distinction here, since I believe that it makes thinking about the homework questions a little more clear. With certain conditions on f, the two will be equal (more or less) at least on the interval $(-\pi, \pi)$.

Here are three (sets of) integrals we are going to need:

$$\int_{-\pi}^{\pi} \cos(kx)\cos(mx)\,dx = \begin{cases} 0 & \text{if } k \neq m \\ 2\pi & \text{if } k = m = 0 \\ \pi & \text{if } k = m \neq 0 \end{cases}$$

$$\int_{-\pi}^{\pi} \sin(kx)\sin(mx)\,dx = \begin{cases} 0 & \text{if } k \neq m \\ \pi & \text{if } k = m \neq 0 \end{cases}$$

$$\int_{-\pi}^{\pi} \sin(kx)\cos(mx)\,dx = 0.$$

They may look like a pain, but with the relevant identities they are easy.

It will not prove what we want to prove, but it is not a difficult calculus problem to show that if there is to be any hope that $F(x) = f(x)$ (the series converges to the given function) then the constants a_k and b_k would necessarily be as follows:

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt.$$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(kt)\,dt \qquad k = 0, 1, 2, \ldots.$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(kt)\,dt \qquad k = 1, 2, 3, \ldots.$$

There is nothing special about using t above instead of x. These integrals are just numbers, after all, and essentially the variable used to describe them is what we might call a "dummy" variable, just there to define the integral. On a later slide, we will want to bring back a_k and b_k in a context where x is already representing something else. Thus, the "t" here to avoid any confusion.

Our version of the theorem requires the ideas of *piecewise continuous* and *piecewise monotone*.

A function is piecewise continuous if on any bounded interval there are only a finite number of points where the function is not continuous.

A function is piecewise monotone if on any bounded interval there are only a finite number of points where the function changes from increasing to decreasing or from decreasing to increasing.

So, here is our version of Dirichlet's Theorem for Fourier series:

Theorem Suppose that f is a function that is bounded, piecewise monotone, and piecewise continuous on the interval $[-\pi, \pi]$. Let

$$F(x) = a_0 + \sum_{k=1}^{\infty} a_k \cos(kx) + b_k \sin(kx)$$

with a_k and b_k defined as above. Then

- F is periodic on $\mathbb{R}$ with period 2π.
- $F(x) = f(x)$ at every $x \in (-\pi, \pi)$ where f is continuous.
- If f is not continuous at $x_0 \in (-\pi, \pi)$, then

$$F(x_0) = \frac{\lim_{x \to x_0^+} f(x) + \lim_{x \to x_0^-} f(x)}{2}$$

Our goal here: let's not attempt to give every detail of the proof, but just get the idea of why these seemingly odd connections with trigonometric functions work out.

The theorem assumes we have function f with domain $[-\pi, \pi]$. In practice, the f in which we are interested might very well have larger domain, perhaps even $\mathbb{R}$. But for purposes of the statement of the theorem, we only care about values on $[-\pi, \pi]$. For the proof of the theorem, however, it would be convenient if our function is defined for all real numbers, but also periodic with period 2π. We could accomplish that by creating a new function by extending the values of f on the interval $[-\pi, \pi)$ repeatedly in both directions. Essentially, we would be making a new function. We could call it something like $\bar{f}$; $\bar{f}$ is the same as f on the original $[-\pi, \pi)$. For example, if $\pi \leq x < 3\pi$, then $\bar{f}(x) = f(x - 2\pi)$, and similarly for every other interval of length 2π. To simplify things, just assume we have already done that, and when we refer to f we are talking about this "extended" periodic function.

And then, a *very* rough outline of the proof:
Partial sum for the F series:

$$F_n(x) = a_0 + \sum_{k=1}^{n} a_n \cos(kx) + b_n \sin(kx).$$

We replace the a_k and b_k with integrals that defined them (using t and dt as the variable in those integrals). These are all finite sums, so we can manipulate the sums and integrals nearly any way we wish, and in doing so end up with $F_n(x)$ equal to an integral from $-\pi$ to π of a function that involves $f(t)$ along with a sum that has products of sine and cosine involving t and x. Some trig identities for those products, more algebra, and we end up with

$$F_n(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} \Big(\frac{1}{2} + \sum_{k=1}^{n} \cos(k(t-x)) \Big) f(t)dt.$$

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Now, a trig identity for the sum of cosines similar to the one for sines that we proved back in section 4.4 (problem 4.4.7):

$$F_n(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{\sin[(2n+1)(t-x)/2]}{2\sin[(t-x)/2]} f(t)dt.$$

Sure, that looks like a mess, but no more "summation" inside the integral. Next, we need to observe that since all of the functions in the integral are periodic with period 2π (assuming we have done our little switch to replace f with a periodic function), it does not matter what interval of length 2π is used for the integral.

$$F_n(x) = \frac{1}{\pi} \int_{-\pi+x}^{\pi+x} \frac{\sin[(2n+1)(t-x)/2]}{2\sin[(t-x)/2]} f(t)dt.$$

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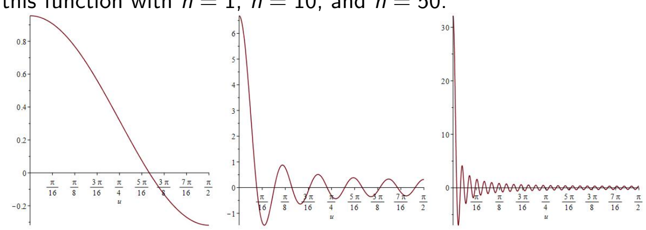
From here, break up the above into two integrals, one from $-\pi + x$ to x and the other from x to $\pi + x$. Then, a substitution, letting $u = -\frac{t-x}{2}$ in the first integral and $u = \frac{t-x}{2}$ in the second. We then have

$$F_n(x) = \frac{1}{\pi} \int_{0}^{\pi/2} \frac{\sin[(2n+1)u]}{\sin[u]} f(x-2u)du + \frac{1}{\pi} \int_{0}^{\pi/2} \frac{\sin[(2n+1)u]}{\sin[u]} f(x+2u)du$$

The behavior of the function $\frac{1}{\pi}\frac{\sin[(2n+1)u]}{\sin[u]}$? Below are graphs of this function with $n = 1$, $n = 10$, and $n = 50$:



The integral of each of those from 0 to $\frac{\pi}{2}$ is $\frac{1}{2}$. But with the "spike" in the graph near $u = 0$, is it clear that as n increases, most of the positive area "weight" in the integral is concentrated in a increasingly narrow interval starting at $u = 0$? And that with the positive and negative fluctuation (and positive/negative canceling) the integral over the remainder of the interval out to $\frac{\pi}{2}$ is getting close to zero?

Thinking about the second of the two integrals defining $F_n(x)$:

$$\frac{1}{\pi}\int_0^{\pi/2} \frac{\sin[(2n+1)u]}{\sin[u]} f(x+2u)du$$

As n gets very large, even with the extra $f(x+2u)$ thrown in, the integral outside of a very (and increasingly) narrow interval starting at zero will get very small, converging to zero If we were carefully writing out the proof, it is at this point we would need the "piecewise monotone" condition. We would not be able to show that this portion of the integral gets small if f was changing direction infinitely often similar to $\frac{\sin[(2n+1)u]}{\sin[u]}$.

So, for any number $\delta > 0$, if n is sufficiently large, then

$$\frac{1}{\pi}\int_0^{\pi/2} \frac{\sin[(2n+1)u]}{\sin[u]} f(x+2u)du \approx \frac{1}{\pi}\int_0^{\delta} \frac{\sin[(2n+1)u]}{\sin[u]} f(x+2u)du$$

And if δ is sufficiently small, then knowing what we know about the continuity f (continuous except at a finite number of points), the values for $f(x+2u)$ will not vary much from the average value of f immediately to the right of x, or approximately

$$\lim_{z\to x^+} f(z).$$

An odd notation, but the text refers to this number as "$f(x+0)$." What we end up with is

$$\frac{1}{\pi}\int_0^{\pi/2} \frac{\sin[(2n+1)u]}{\sin[u]} f(x+2u)du \approx f(x+0)\frac{1}{\pi}\int_0^{\delta} \frac{\sin[(2n+1)u]}{\sin[u]} du$$

$$\approx f(x+0)\cdot\frac{1}{2}$$

We deal with the other integral for $F_n(x)$ in the same way, except in this case we have $\lim_{z\to x^-} f(z)$. Letting $n\to\infty$, our "$\approx$" turn into "$=$" and we end up with

$$F_n(x) \to F(x) = \frac{1}{2}\cdot \lim_{z\to x^-} f(z) + \frac{1}{2}\cdot \lim_{z\to x^+} f(z).$$

And of course for values of x where f is continuous, the limits from left and right are both equal to $f(x)$, and our theorem is complete.