MATH 510, Just a few things not otherwise discussed!

Modern Analysis

James Madison University

In the (complete) proof of Dirichlet's theorem, the idea of *uniform continuity* is introduced. This is not a great leap, since it is parallel to the other "uniform" concepts. A function is continuous at point a if given $\epsilon > 0$ there is a corresponding $\delta > 0$ so that $|f(x) - f(a)| < \epsilon$ when $|x - a| < \delta$. It should not be surprising that uniform continuity on an interval just means that for a given ϵ we can find a single δ that works at every point on that interval.

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There is a more general version of this theorem, requiring that the domain be *compact* (see the next couple slides for what this means). Earlier theorems about continous functions whose domain is a closed interval being bounded and actually hitting their max and min values are also true more generally if the domain is a compact set.

So far, we have had no compelling reason to discuss the ideas of open or closed sets. Of course everyone knows what we mean by an open interval or closed interval; we are just generalizing that. In $\mathbb{R}$, a set is *open* if it is the union of a collection of open intervals. Possibly an empty union, so that by definition the empty set $\emptyset$ is open.

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- Finite sets are closed.
- $\mathbb{R}$ and $\emptyset$ are both open and closed. The only sets that are both open and closed.

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For the general definition of compact and how it works with the real numbers (or more generally in $\mathbb{R}^n$) you would need to explore the *Heine Borel Theorem*.

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Further complicating this issue is the question of how the integral is defined. In the early 20th century, an alternative to the Riemann integral was introduced by Henri Lebesgue. The class of functions that are *Lebesgue integrable* is strictly larger than the class of functions that is *Riemann integrable*.

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Just one definition from measure theory: We say that set $A \subseteq \mathbb{R}$ has measure zero if given any number $\epsilon > 0$ it is possible to find a collection of open intervals $\{I_n\}$ such that

$$A \subseteq \cup_{n=1}^{\infty} I_n \text{ and } \sum_{n=1}^{\infty} \ell(I_n) < \epsilon$$

(where $\ell(I)$ is the length of interval I).

Just a couple theorems we are not going to prove:

Theorem Suppose that $\{f_n\}$ is a sequence of functions, $f_n \to f$ pointwise on the interval $[a, b]$, $|f_n(x)| \leq g(x)$ for $x \in [a, b]$, and that f, g and each f_n is integrable. Then

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