MATH 510, Just a few things not otherwise discussed!

Modern Analysis

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In the (complete) proof of Dirichlet's theorem, the idea of *uniform continuity* is introduced. This is not a great leap, since it is parallel to the other "uniform" concepts. A function is continuous at point a if given $\epsilon > 0$ there is a corresponding $\delta > 0$ so that $|f(x) - f(a)| < \epsilon$ when $|x - a| < \delta$. It should not be surprising that uniform continuity on an interval just means that for a given ϵ we can find a single δ that works at every point on that interval.

A following theorem (described there as merely a "Lemma") says that a function that is continuous on a closed and bounded interval is uniformly continuous.

There is a more general version of this theorem, requiring that the domain be *compact* (see the next couple slides for what this means). Earlier theorems about continous functions whose domain is a closed interval being bounded and actually hitting their max and min values are also true more generally if the domain is a compact set.

So far, we have had no compelling reason to discuss the ideas of open or closed sets. Of course everyone knows what we mean by an open interval or closed interval; we are just generalizing that. In $\mathbb{R}$, a set is *open* if it is the union of a collection of open intervals. Possibly an empty union, so that by definition the empty set $\emptyset$ is open. A set is *closed* if its complement is open.

A few facts:

- Set A is closed *iff* for every convergent sequence $\{x_n\} \subseteq A$ the limit of the sequence is also in A.
- The union of any collection of open set is open. The intersection of any collection of closed sets is closed.
- Finite sets are closed.
- $\mathbb{R}$ and $\emptyset$ are both open and closed. The only sets that are both open and closed.

These are ideas from *topology*, where concepts of open, closed, continuous, etc. are extended to more general *topological spaces*.

We will not attempt to provide the more general definitions used in topology. Suffice it to say that some of them need to be stated in a manner different than in $\mathbb{R}$, but those definitions are equivalent to the more familiar concepts in $\mathbb{R}$.

Compact in particular is a tricky concept to handle in more general topological spaces, but in $\mathbb{R}$ the compact sets are the ones that are closed and bounded.

For the general definition of compact and how it works with the real numbers (or more generally in $\mathbb{R}^n$) you would need to explore the *Heine Borel Theorem*.

Integrals: Nearly every course in calculus introduces the (Riemann) integral in a reasonable way. Most of section 6.2 and 6.3 is a readable review of all that.

Treatment of integrals often begins with bounded function defined on bounded intervals, and then is extended to the idea of so-called *improper intgrals* where the function and/or the interval of integration is not bounded.

The question of whether an integral exists, that is, whether or not a function is *integrable* can be a little trickier than the corresponding question for derivatives. Likewise, the question of when the limit of a sequence of integrable functions is integrable can be problematic.

Further complicating this issue is the question of how the integral is defined. In the early 20th century, an alternative to the Riemann integral was introduced by Henri Lebesgue. The class of functions that are *Lebesgue integrable* is strictly larger than the class of functions that is *Riemann integrable*.

Theorems about convegence of integrals hold more generally. Lebesgue integration is closely tied to the development of *measure theory*, which in turn is closely linked to the development of better theoretical approaches to *probability theory*.

Just one definition from measure theory: We say that set $A \subseteq \mathbb{R}$ has measure zero if given any number $\epsilon > 0$ it is possible to find a collection of open intervals $\{I_n\}$ such that

$$A \subseteq \cup_{n=1}^{\infty} I_n \text{ and } \sum_{n=1}^{\infty} \ell(I_n) < \epsilon$$

(where $\ell(I)$ is the length of interval I).

Just a couple theorems we are not going to prove:

Theorem Suppose that $\{f_n\}$ is a sequence of functions, $f_n \to f$ pointwise on the interval $[a, b]$, $|f_n(x)| \leq g(x)$ for $x \in [a, b]$, and that f, g and each f_n is integrable. Then

$$\int_a^b f_n \to \int_a^b f.$$

Theorem Function f is Riemann integrable on $[a, b]$ *iff* f is continous on $[a, b]$ except on a set of measure zero.

By the way, when some property holds except for a set of measure zero, we say that the property holds *almost everywhere*. No joke. *"a.e."*

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