MATH 510, Sequences and Subsequences

Modern Analysis

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First, a couple things about sequences not previously mentioned:

Given any sequence

$$\{s_n\} = \{s_1, s_2, s_3, \cdots\}$$

we say that $\{s_n\}$ is *increasing* if $s_n > s_m$ when $n > m$.

If we replace $s_n > s_m$ with $s_n < s_m$ in the above, the sequence is *decreasing*. If instead we only know $s_n \geq s_m$ we have the somewhat confusing term *non-decreasing* (perhaps "never-decreasing" would be better). If $s_n \leq s_m$ then *non-increasing*.

The generic term covering all of the above is *monotone*. A sequence is monotone if it never changes direction. Perhaps it is clear what we would mean if we said a sequence is "eventually monotone"?

Just for the record, a sort-of obvious theorem (using the least upper bound property):

Theorem Every bounded monotone sequence converges.

Increasing sequences converge to the least upper bound, etc.
All of the above was sort of implicit when we were introducing the ideas of limit superior, limit inferior. The U_n and L_n sequences introduced then were monotone.

And also: one reason we needed to think about absolute convergence of series is the fact that for series with all positive terms, the sequence of partial sums is monotone.

Given any sequence

$$\{s_n\} = \{s_1, s_2, s_3, \cdots\}$$

a *subsequence* of $\{s_n\}$ is a new sequence consisting of an infinite number of terms from the original sequence, but necessarily in the same order.

Note that "an infinite number" of the original terms could mean "all," so that technically every sequence is a subsequence of itself (just as every set is a subset of itself).

$$\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots\},$$

$$\{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16} \cdots\}$$

$$\{1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \cdots\}.$$

Above, the second and third sequences are *subsequences* of the first.
Is it clear that the following *not* subsequences of the first sequence?

$$\{1, \frac{1}{5}, \frac{1}{3}, \frac{1}{9}, \frac{1}{7}, \cdots\}$$

$$\{1, \frac{1}{2}, \frac{2}{3}, \frac{1}{4}, \frac{4}{5}, \cdots\}$$

$$\{1, \frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \cdots\}$$

Inventing a notation to describe this: Maybe not always absolutely neessary, but it would be needed to write out formal proofs of theorems involving subsequences: Does the following capture the idea of subsequence?

Let $\omega : \mathbb{N} \to \mathbb{N}$ be a *strictly* increasing function. That is, if $n < m$ this would imply $\omega(n) < \omega(m)$.

Then given sequence $\{s_n\}$, $\{s_{\omega(n)}\}$ is a *subsequence* of s_n.

A theorem about subsequences, not difficult to prove:

Theorem Given a sequence $\{s_n\}$, $s_n \to L$ *iff* every subsequence of $\{s_n\}$ converges to L.

Somewhat more difficult to prove, but not too daunting:

Theorem (Bolzano-Weierstrauss) Every bounded sequence has a subsequence that converges.

The proof can go a couple different ways. The following is just an outline, of course: A somewhat direct approach uses the nested interval property. This involves starting with an interval in the real numbers that includes an upper bound and a lower bound for the sequence. Cut that interval in half, note that at least one of the resulting two intervals has an infinite number of terms from the original sequence. Cut that interval in half, and again choose a subinterval containing an infinite number of terms from the original sequence. Repeating this process leads to a collection of intervals having the nested interval property. One may then choose a term from the sequence from each of those intervals, always taking a larger "n" than the previous term, to obtain a convergent subsequence.

For an alternate proof using the least upper bound property, one could begin by tackling the following:

lemma Every sequence has a subsequence that is monotone.

The Bolzano-Weierstrauss theorem is sometimes stated other ways. One possibility uses the notion of an "accumulation point" (look that up if you wish).

In any case, B-W is just one more alternate form characterizing the idea of completeness in the real numbers. There is no B-W theorem for the rational numbers, for example.