Assignment Math 424

Two contestants, Statistician and Nature, put up 1 or 2 fingers simultaneously. The loss to the statistician is $L(\theta, a) = (\theta - a)^2$.

	Nature $\theta = 1$	Nature $\theta = 2$
Statistician $a = 1$	0	1
Statistician $a = 2$	1	0

1. If it is known $P(\theta = 1) = 1/3, P(\theta = 2) = 2/3$. What is the best action for the statistician?

Take $a = 2$.

2. Suppose instead putting up his fingers, Statistician is allowed to put a real number on a piece of paper. What number should he present?

Answer: $a = E(\theta) = 5/3$.

Now suppose Statistician can ask the Nature how many fingers he will put up. Let X= number of fingers Nature tells Statistician he will put up. The probability distribution of X is given below.

$\theta = 1$: $P(X = 1) = 3/4, P(X = 2) = 1/4$.

$\theta = 2$: $P(X = 2) = 3/4, P(X = 1) = 1/4$.

Assume Statistician can choose a real number. What is his best action if $X = 1$ is observed? If $X = 2$ is observed?

If $X = 1$ is observed, the conditional distribution of θ is

$P(\theta = 1|X = 1) = \frac{P(X=1|\theta=1)P(\theta=1)}{P(X=1|\theta=1)P(\theta=1)+P(X=1|\theta=2)P(\theta=2)} = \frac{\frac{3}{4}*\frac{1}{3}}{\frac{3}{4}*\frac{1}{3}+\frac{1}{4}*\frac{2}{3}} = \frac{3}{5}$

and $P(\theta = 2|X = 1) = 2/5$.

so the Bayes action is $a = E(\theta|X = 1) = 1 * 0.6 + 2 * 0.4 = 1.4$.

If $X = 2$ is observed, the conditional distribution of θ is

$P(\theta = 1|X = 2) = \frac{P(X=2|\theta=1)P(\theta=1)}{P(X=2|\theta=1)P(\theta=1)+P(X=2|\theta=2)P(\theta=2)} = \frac{\frac{1}{4}*\frac{1}{3}}{\frac{1}{4}*\frac{1}{3}+\frac{3}{4}*\frac{2}{3}} = \frac{1}{7}$

and $P(\theta = 2|X = 2) = \frac{6}{7}$.

The Bayes action is $a = E(\theta|X = 2) = 1 * \frac{1}{7} + 2 * \frac{6}{7} = 13/7$.

Assume again Statistician can choose a real number and his loss function is given by $L(\theta, a) = \theta(\theta - a)^2$. What is his best action if $X = 1$ is observed? If $X = 2$ is observed? (You can write your answers on the back)

$f(a) = EL(\theta, a) = E\theta(\theta - a)^2 = E(\theta^3 - 2\theta^2 a + \theta a^2) = E(\theta^3) - 2aE(\theta^2) + a^2 E(\theta)$,

$f'(a) = -2E(\theta^2) + 2aE(\theta) = 0$,

Hence $a = \frac{E(\theta^2)}{E(\theta)}$.

If $X = 1$ is observed, the conditional distribution of θ is
$P(\theta = 1|X = 1) = 0.6, P(\theta = 2|X = 1) = 0.4$,
and $E(\theta^2|X = 1) = 1^2 * 0.6 + 2^2 * 0.4 = 2.2$,
$E(\theta|X = 1) = 1 * 0.6 + 2 * 0.4 = 1.4$,
and $a = \frac{2.2}{1.4} = 11/7$.
Similarly if $X = 2$ is observed, the Bayes action is
$a = 25/13$.