Introduction to Probability

experiment–any action or process whose outcome is uncertain.
sample space– the set of all possible outcomes of an experiment.
notation: $\mathcal{S}$.
event–a subset of outcomes in the sample space.
notation: A, B, C
union: A or B. $A \cup B$, contains outcomes either in A or B or both.
intersection: A and B. $A \cap B$, contains outcomes both in A and B.
complement of an event A: outcomes in $\mathcal{S}$ but not in A.
notation: A' or A^c.
example: flip a coin twice.
$\mathcal{S}$=$\{HH, HT, TH, TT\}$.
A=first flip is H=$\{HH, HT\}$.
B=second flip is T=$\{HT, TT\}$.
$A \cup B = \{HH, HT, TT\}$.
$A \cap B = \{HT\}$.
$A' = \{TH, TT\}$.

Axioms and properties of probability

Axiom 1 For any event, $P(A) \geq 0$.
Axiom 2 $P(\mathcal{S})$=1.
Axiom 3 If $A_1, A_2, \cdots$, is an infinite collection of disjoint events, then $P(A_1 \cup A_2 \cup A_3 \cup \cdots) = \sum_{i=1}^{\infty} P(A_i)$.

proposition:

- $P(\phi) = 0$, where ϕ is an empty set.
- $P(A) = 1 - P(A')$.
- For any events A and B,
  $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

Example

In Harrisonburg, 60% of households subscribe to NY Times, 45% subscribe to Washington Post, and 25% subscribe to both. What is the probability that a randomly selected household subscribe to (1) at least one of the two newspapers (2) exactly one of the two newspapers?

Answer

P(subscribe to at least one)
$=P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.6 + 0.45 - 0.25 = 0.8.$

P(subscribe to exactly one)
$=P(A \cup B) - P(A \cap B) = 0.8 - 0.25 = 0.55.$

Exercise

Near a certain exit of I-81, for a truck stopped at a roadblock, the probability is 0.23 that it will have faulty brakes, the probability is 0.24 that it will have badly worn tires. Also the probability is 0.38 that it will have faulty brakes and/or badly worn ties. What is the probability that a truck stopped here will have both faulty brakes and badly worn tires?

Conditional probability

The **conditional probability** of A given that B has occurred is $P(A|B) = \frac{P(A \cap B)}{P(B)}$.
Example:

	BigEater	NotBigEater	total
Affluent	0.2	0.1	0.3
Not Affluent	0.2	0.5	0.7
total	0.4	0.6	1.0

Table: Probability table for customers among various classes

Find $P(BigEater|Affluent), P(Affluent|NotBigEater)$.
$P(BE|A) = \frac{P(BE \text{and} A)}{P(A)} = 0.2/0.3 = 2/3.$
$P(A|NotBE) = P(A\text{and } NotBE)/P(NotBE) = 0.1/0.6 = 1/6..$

Multiplication rule

$P(A \cap B) = P(A|B)P(B) = P(B|A)P(A)$.
There are 4 red balls and 3 black balls of the same size in a jar. Randomly take out 1 ball, put it aside, then randomly take out another ball.
(Note this is called Sampling without replacement. Sampling with replacement means you put the ball back each time.)
Find the probability that both balls are red.
$P(\text{1st Red and 2nd Red}) = P(\text{2nd Red}|\text{1st Red}) * P(\text{1st Red}) = (3/6) * (4/7) = 2/7$.

Bayes$'$ theorem

The law of total probability: Let $A_1, \cdots, A_k$ be mutually exclusive and exhaustive events (or a partition of $\mathcal{S}$).
Then for any other event B,
$P(B) = P(B|A_1)P(A_1) + P(B|A_2)P(A_2) + \cdots + P(B|A_k)P(A_K) = \sum_{i=1}^{k} P(B|A_i)P(A_i)$.

Example

Suppose you will be guided into room A, B, C with probability 0.5, 0.25, 0.25 respectively. The probability that you will see a red light in room A is 0.2, in room B is 0.5, in room C is 0.1. What is the probability that you will see a red light?
$P(red) = P(red|A)P(A) + P(red|B)P(B) + P(red|C)P(C) = 0.2 * 0.5 + 0.5 * 0.25 + 0.1 * 0.25 = 0.25.$

Bayes' theorem

Let $A_1, \cdots, A_k$ be mutually exclusive and exhaustive events with $P(A_i) > 0$ for $i = 1, 2, \cdots, k$. Then for any other event B for which $P(B) > 0$,

$P(A_j|B) = \frac{P(A_j \cap B)}{P(B)} = \frac{P(B|A_j)P(A_j)}{\sum_{i=1}^{k} P(B|A_i)P(A_i)}$.

In a certain community, 1% of adults have diabetes. A hospital in this community correctly diagnoses 95 percent of all persons with diabetes as having the disease and incorrectly diagnoses 2 percent of all persons without diabetes as having the disease. Find the probability that an adult diagnosed by the hospital as having diabetes actually has the disease.

Let S indicate sick (actually having diabetes) and D diagnosed as having diabetes, H indicate healthy, then

$$P(S|D) = \frac{P(D|S)P(S)}{P(D|S)P(S)+P(D|H)P(H)} = \frac{0.95*0.01}{0.95*0.01+0.02*0.99} = 0.32$$

exercise

Exercise: There are three bags labeled 1, 2, and 3. These bags contain respectively 3 white and 3 black balls, 4 white and 2 black balls, and 1 white and 2 black balls. Select one bag randomly and then draw a ball randomly from this bag. Given that a black ball is drawn, what is the probability that bag 2 had been selected?

Exercise

In a certain state, 25% of all cars emit excessive amounts of pollutants. If the probability is 0.99 that a car emitting excessive amounts of pollutants will fail the state's emission test, and the probability is 0.17 that a car not emitting excessive amounts of pollutants will nevertheless fail the test, what is the probability that a car which fails the test actually emits excessive amounts of pollutants?

Solution

$$P(E|F) = \frac{P(F|E)P(E)}{P(F|E)P(E)+P(F|NE)P(NE)} = \frac{0.99*0.25}{0.99*0.25+0.17*0.75} = 0.66.$$

A mail order house has three stock clerks, U, V, and W, who pull items from shelves and assemble them for packaging. U makes a mistake in an order one time in a hundred, V makes a mistake five times in a hundred, and W makes a mistake three times in a hundred. U, V, and W fill, respectively, 30, 40 and 30 percent of all orders.

a). Find the probability that a mistake will be made in an order.

b). If a mistake is made in an order, find the probability that it was filled by V.

Independence

Two events A and B are independent if $P(A|B) = P(A)$ or $P(B|A) = P(B)$
or $P(A \cap B) = P(A)P(B)$
and dependent otherwise.
Independence of more than two events.
Events $A_1, A_2, \cdots, A_n$ are mutually independent if for every $k (k = 1, 2, 3, \cdots, n)$ and every subset of indices $i_1, i_2, \cdots, i_k$
$P(A_{i_1} \cap A_{i_2} \cdots \cap A_{i_k}) = P(A_{i_1})P(A_{i_2}) \cdots P(A_{i_k})$.
Example: A, B, C are mutually independent if
$P(A \cap B) = P(A)P(B), P(A \cap C) = P(A)P(C), P(B \cap C) = P(B)P(C)$ and $P(A \cap B \cap C) = P(A)P(B)P(C)$.

A shooter hits a target with probability 0.75. Assuming independence, find the probability

a). All hits in 3 shoots.

b). a hit followed by two misses.

Let H_i, M_i be a hit (miss) on the ith shoot, then

a). $P(\text{all hits}) = P(H_1 \cap H_2 \cap H_3) = P(H_1)P(H_2)P(H_3) = 0.75^3 = 0.422.$

b).

$P(H_1 \cap M_2 \cap M_3) = P(H_1)P(M_2)P(M_3) = 0.75 * 0.25 * 0.25 = 0.047.$