Testing hypothesis

Any statement about the unknown parameter θ is called a hypothesis.
If a hypothesis completely specifies the distribution of $X_1, \cdots, X_n$, then it is called a **simple hypothesis**. Otherwise it is called a **composite hypothesis**.
e.g., $X_1, \cdots, X_n \sim N(\theta, 1)$.
Simple hypothesis: $\theta = 0$, or $\theta = 2$
Composite hypothesis: $\theta > 0$.
Test function $\phi(x_1, \cdots, x_n)$ is a mapping from the sample space $\mathcal{X}$ into [0, 1]. If we observe $(x_1, \cdots, x_n)$, then we reject H_0 with probability $\phi(x_1, \cdots, x_n)$.
For simplicity, write $\underline{x} = (x_1, x_2, \cdots, x_n)$
when $\phi(\underline{x}) \in \{0, 1\}$, the ϕ is called a **simple test function**.
$R_\phi = \{x : \phi(\underline{x}) = 1\}$: **rejection region (critical region).**
$A_\phi = \{x : \phi(\underline{x}) = 0\}$: **acceptance region.**

Type I and Type II error

Type I and Type II error.
Type I error: rejects H_0 when H_0 is true.
Type II error: rejects H_1 when H_1 is true.

Size and Power of a test

$H_0 : \theta \in \Theta_0$
$H_1 : \theta \in \Theta_1$

Probability of Type I error:
For $\theta \in \Theta_0 : P_\theta(\phi(\underline{x}) = 1) = \mathrm{E}_\theta(\phi(\underline{x}))$.

Probability of Type II error:
For $\theta \in \Theta_1 : P_\theta(\phi(\underline{x}) = 0) = 1 - \mathrm{E}_\theta(\phi(\underline{x}))$.

Size of a test: $\max_{\theta \in \Theta_0} \mathrm{E}_\theta \phi(\underline{x})$. It is also called **the level of significance**.
Power function: $\pi_\phi(\theta) = \mathrm{E}_\theta \phi(\underline{x})$ is called the power function.
If $\theta \in \Theta_0$, $\pi_\phi(\theta) =$ prob. of Type I error.
If $\theta \in \Theta_1$, $\pi_\phi(\theta) = 1 -$ prob. of Type II error.

The size of a test is the maximum of prob. of Type I error.
We want $\pi_\phi(\theta)$ to be large when $\theta \in \Theta_1$ and small when $\theta \in \Theta_0$.

Usually Type I and Type II error can not be made small at the same time when n is fixed. Traditionally, we control the prob. of Type I error and minimize the prob. of Type II error.

Example

Let $X \sim N(\theta, 1)$.
$H_0 : \theta \geq 1$ vs $H_1 : \theta < 1$.
Consider the test function $\phi(x) = 1$ if $x < 1$ and $\phi(x) = 0$ if $x \geq 1$. i.e., reject H_0 if we observe $x < 1$ and accept H_0 if we observe $x \geq 1$.
Find the power function of ϕ.
Find the prob. of Type I error when $\theta = 1$.
Find the size of the test.
Find the prob. of Type II error when $\theta = 0.5$.
Repeat the problem for $\phi(\underline{x}) = 1$ if $\bar{x}_n < 1$ for $n = 25$.

Solutions

Power function $\pi_\phi(\theta) = P(X < 1)$.
Prob. of Type I error $= P(X < 1; \theta = 1) = P(Z < 0) = 0.50$.
Size of the test $= \max_{\theta \geq 1} P(X < 1; \theta) = P(Z < 0) = 0.50$.
That is, for $\theta \geq 1$, $\theta = 1$ maximizes the probability of Type I error.
Prob. of Type II error $=$
$P(X \geq 1; \theta = 0.5) = P(Z > 0.5) = 0.3085$

For $n = 25$.
Power function $P(\bar{X}_n < 1)$.
Prob. of Type I error$= P(\bar{X}_n < 1; \theta = 1) = P(Z < 0) = 0.5$.
Size of the test $=\max_{\theta \geq 1} P(\bar{X}_n < 1; \theta) = P(\bar{X}_n < 1; \theta = 1) = 0.5$.
Prob. of Type II error $=$
$P(\bar{X}_n \geq 1; \theta = 0.5) = P(Z > 2.5) = 0.0062$.

Bayes test

The Bayes test minimizes a Bayes risk under an appropriate loss function.
Test $H_0 : \theta \in \Theta_0$ vs $H_1 : \theta \in \Theta_0^c$.
Let $a_0 =$ accept $H_0, a_1 =$ accept H_1. Then
$L(\theta, a_0) = 1$ if $\theta \in \Theta_0^c$
$L(\theta, a_0) = 0$ otherwise.
$L(\theta, a_1) = 1$ if $\theta \in \Theta_0$
$L(\theta, a_1) = 0$ otherwise.
This is called $0 - 1$ **loss function**.
You can imagine you guess the value of θ, you loss is 0 if your guess is right and your loss is 1 if your guess is wrong.

Bayes test

Posterior distribution of θ
$g(\theta|\underline{x}) = \frac{f(\underline{x};\theta)g(\theta)}{\int f(\underline{x};\theta)g(\theta)d\theta}$.
The posterior prob. of Θ_0 is
$P(\theta \in \Theta_0|\underline{x}) = \int_{\Theta_0} g(\theta|\underline{x})d\theta$
The posterior prob. of Θ_0^c is
$1 - P(\theta \in \Theta_0|\underline{x})$.
The Bayes test under the 0-1 loss function is
$\phi^*(\underline{x}) = 1$ if $P(\theta \in \Theta_0^c|\underline{x}) \geq P(\theta \in \Theta_0|\underline{x})$
and $\phi^*(\underline{x}) = 0$ otherwise

Example

$X_1, \cdots, X_n$iid, $\sim$ Bernoulli (θ).
Test $H_0 : \theta = 0.2$ vs $H_1 : \theta = 0.8$.
The prior distribution of θ is $P(\theta = 0.2) = P(\theta = 0.8) = 0.5$. Find the Bayes test under the 0-1 loss function.

write $\underline{X}_n = X_1, \cdots X_n$
Reject H_0 if $P(\theta = 0.8 | \underline{x}_n) > P(\theta = 0.2 | \underline{x}_n)$.
Or $P(\underline{X}_n | \theta = 0.8) P(\theta = 0.8) > P(\underline{X}_n | \theta = 0.2) P(\theta = 0.2)$
or $P(\underline{X}_n | \theta = 0.8) > P(\underline{X}_n | \theta = 0.2)$
Now $P(\underline{X}_n | \theta = 0.8) = 0.8^{\sum x_i} 0.2^{n - \sum x_i}$
and $P(\underline{X}_n | \theta = 0.2) = 0.2^{\sum x_i} 0.8^{n - \sum x_i}$
(note for Bernoulli r.v., $P(X_i = x_i | \theta) = \theta^{x_i} (1 - \theta)^{1 - x_i}$)
Let $\frac{P(\underline{X}_n | \theta = 0.8)}{P(\underline{X}_n | \theta = 0.2} > 1$
we can solve
reject H_0 if $\sum X_i > n/2$.

Example

Let $X_1, \cdots, X_n$ iid Bernoulli (θ). The prior distribution of θ is $g(\theta) \sim Uniform(0, 1)$.
Test $H_0 : \theta \leq 1/2$ vs $H_1 : \theta > 1/2$. Find the Bayes test under 0-1 loss.
Note the posterior distribution for θ,
$g(\theta|\underline{x})$ is a beta distribution with parameters $(\alpha = \sum x_i + 1, \beta = n - \sum x_i + 1)$,
so the Bayes test is to
Reject H_0 if $\int_{1/2}^{1} beta(\theta; \sum x_i + 1, n - \sum x_i + 1) d\theta > 1/2$.

Bayes test exercise

$X_1, \cdots, X_n \sim$ Bernoulli (θ).
Test $H_0 : \theta = 0.2$ vs $H_1 : \theta = 0.7$.
The prior distribution of θ is $P(\theta = .2) = 0.4, P(\theta = 0.8) = 0.6$.
Find the Bayes test under the 0-1 loss function.