Math 424 Exercises

1. Let $X_1, \cdots, X_n$ iid $\sim N(\theta, 1)$.
Consider two estimators of θ.
$T_1 = \bar{X}$ and $T_2 = \frac{1}{2}(X_1 + X_2)$.
Find the MSE of both estimators. Are they consistent?
Note both T_1 and T_2 are unbiased for θ,
so $E(T_1 - \theta)^2 = V(T_1) = \frac{\sigma^2}{n} = \frac{1}{n}$.
As $\frac{1}{n} \to 0$, T_1 is (MSE) consistent.
$V(T_2) = V(\frac{1}{2}X_1 + \frac{1}{2}X_2) = \frac{1}{4}V(X_1) + \frac{1}{4}V(X_2) = \frac{1}{4} * 1 + \frac{1}{4} * 1 = 1/2$.
So T_2 is not (MSE) consistent as its MSE does not tend to 0.

2. Let X be one observation from the $Bin(n, \theta)$ distribution where n is an known constant. Consider two estimators for θ.
$T_1 = \frac{X}{n}$ and $T_2 = \frac{X+2}{n+4}$.
1).Are T_1 and T_2 unbiased for θ?
$E(T_1) = E(\frac{X}{n}) = \frac{1}{n}E(X) = \frac{1}{n}(n\theta) = \theta$, so T_1 is unbiased for θ.
$E(T_2) = E(\frac{X+2}{n+4)}) = \frac{1}{n+4}E(X+2) = \frac{E(X)+2}{n+4} = \frac{n\theta+2}{n+4} \neq \theta$ unless $\theta = 1/2$.
So $E(T_2)$ is biased for θ.
2). Find the MSE of T_1 and T_2.
$E(T_1 - \theta)^2 = V(T_1) = \frac{1}{n^2}V(X) = \frac{1}{n^2}(n\theta(1-\theta)) = \frac{\theta(1-\theta)}{n}$.
Bias of T_2 : $E(T_2) - \theta = E(\frac{X+2}{n+4}) - \theta = \frac{n\theta+2}{n+4} - \theta = \frac{2-4\theta}{n+4}$.
variance of T_2: $V(T_2) = (\frac{1}{n+4})^2 V(X) = \frac{n\theta(1-\theta)}{(n+4)^2}$,
MSE of T_2 : $E(T_2 - \theta)^2 = (\frac{2-4\theta}{n+4})^2 + \frac{n\theta(1-\theta)}{(n+4)^2}$