The Poisson distribution

A random variable is said to have a **Poisson distribution** with parameter $\lambda (\lambda > 0)$ if the pmf of X is

$p(X = k) = \frac{e^{-\lambda}\lambda^k}{k!}, k = 0, 1, 2, 3, \cdots.$

check $\sum_{k=0}^{\infty} \frac{e^{-\lambda}\lambda^k}{k!} = e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = e^{-\lambda} e^{\lambda} = 1.$

The mgf of X is

$M_X(t) = \sum_{k=0}^{\infty} e^{tk} \frac{e^{-\lambda}\lambda^k}{k!}$

$= e^{-\lambda} \sum_{k=0}^{\infty} \frac{(\lambda e^t)^k}{k!} = e^{-\lambda} e^{\lambda e^t} = e^{\lambda(e^t - 1)}.$

$M_X^{(1)}(t) = \lambda e^t e^{\lambda(e^t - 1)},$

$M_X^{(2)}(t) = \lambda e^t e^{\lambda(e^t - 1)} + \lambda^2 e^{2t} e^{\lambda(e^t - 1)}.$

$E(X) = \lambda, Var(X) = \lambda.$

Example: Let X be the number of creatures of a certain type captured in a trap during a day. Suppose X has a Poisson distribution with $\lambda = 4.5$. i.e., on average the trap will capture 4.5 creatures a day. The probability that a trap capture 5 creatures in a day is

$P(X = 5) = \frac{e^{-4.5}(4.5)^5}{5!} = 0.1708.$

The probability that the trap will capture 0 creatures is

$P(X = 0) = \frac{e^{-4.5}(4.5)^0}{0!} = e^{-4.5} = 0.011.$

The probability that the trap will capture 12 creatures is

$P(X = 12) = \frac{e^{-4.5}(4.5)^{12}}{12!} = 0.0016.$

approximating binomial distribution

proposition: The binomial pmf approaches the Poisson pmf if we let $n \to \infty, p \to 0$ in such a way that $np \to \lambda > 0$.
proof p 223.

example

a typesetter, on average, makes one error in every 500 words typeset. A typical pages contains 300 words. What is the probability that there will be no more than 2 errors in 5 pages?
Let X be the number of errors in 5 pages, then $X \sim$ binomial $(1500, \frac{1}{500})$, and
$P(X \leq 2) = \sum_{x=0}^{2} \binom{1500}{x} (\frac{1}{500})^x (\frac{499}{500})^{1500-x} = 0.4230.$
If we use $X \sim$ Poisson with $\lambda = np = 3$, then
$P(X \leq 2) = e^{-3}(1 + 3 + 3^2/2) = 0.4232.$

The Law of small numbers

The Poisson distribution can serve for number of events during a given time interval or in a specified region when
1). The events in nonoverlapping time intervals or regions are independent.
2). The probability of a single event occurring in a very small time interval or region is proportional to the length of the time interval or size of the region.
3). The probability of more than one events occurring a such a short time interval or a small region is negligible.

exercise

A telephone operator on averages handles 5 calls every 3 minutes. What is the probability that there will be no calls in the next minute?

0.189.

Poisson/Exponential relationship

Theorem: Suppose a series of events satisfying the Poisson model are occurring at the rate of λ per unit time. Let the random varible Y be the interval between consecutive events. Then Y has the exponential distribution $f_Y(y) = \lambda e^{-\lambda}, y > 0$.

proof: Suppose an event occurred at time a. Then the probability that no event will occur in $(a, a+y)$ is $\frac{e^{-\lambda y}(\lambda y)^0}{0!} = e^{-\lambda y}$.
Note no event in $(a, a+y)$ iff $Y > y$. So
$P(Y > y) = e^{-\lambda y}$ and $F_Y(y) = 1 - e^{-\lambda y}$, $f_Y(y) = \lambda e^{-\lambda y}, y > 0$.

Example

4.2.26. Suppose the commercial airplane crashes in a certain country occur at the rate of 2.5 per year. What is the probability that the next two crashes will occur within 3 months of one another?
The rate 2.5 per year is the same as 2.5/12 per month.
The time between the next two crashes $\sim$ exponential
$\lambda = 2.5/12 = 0.2083$
$P(Y < 3) = F_Y(3) = 1 - e^{-0.2083*3} = 0.465.$

Exercise

Records show that deaths occur at the rate of 0.1 per day among patients in a large nursing home. If someone dies today, what are the chances that a week or more will elapse before another death occurs?
$e^{-0.7} = 0.50$.