Method of Moments Estimator

Population moments:
$\mu_j = E(X^j)$, the j-th moment of X.
Sample moments:
$m_j = \frac{1}{n}\sum_{i=1}^{n} X_i^j$.
e.g, j=1, $\mu_1 = E(X)$, population mean
$m_1 = \bar{X}$: sample mean.

MME estimators of $\theta_1, \cdots, \theta_k$, defined $\hat{\theta}_1, \cdots, \hat{\theta}_k$, are defined as solutions to the system of equations:

$\mu_1(\hat{\theta}_1, \cdots, \hat{\theta}_k) = m_1$

$\mu_2(\hat{\theta}_1, \cdots, \hat{\theta}_k) = m_2$

...

$\mu_k(\hat{\theta}_1, \cdots, \hat{\theta}_k) = m_k$.

example:
$X_1, \cdots, X_n$, iid $\sim N(\mu, \sigma^2), \theta = (\mu, \sigma^2)$.
$\mu_1(\theta) = \mu$,
$\mu_2(\theta) = \mu^2 + \sigma^2$.
MM equations:
$\hat{\mu} = \bar{X}_n$,
$\hat{\mu}^2 + \hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^n X_i^2$
Solutions:
$\hat{\mu} = \bar{X}_n$,
$\hat{\sigma}^2 = \frac{1}{n}\sum X_i^2 - \bar{X}_n^2 = \frac{1}{n}\sum(X_i - \bar{X}_n)^2$.

Example

Suppose that $Y_1 = 0.42$, $Y_2 = .10$, $Y_3 = 0.65$, and $Y_4 = 0.23$ is a random sample of size 4 from the pdf $f_Y(y;\theta) = \theta y^{\theta-1}, 0 \leq y \leq 1$.
Find the MME for θ.
$E(Y) = \int_0^1 y\theta y^{\theta-1} dy = \frac{\theta}{\theta+1}$.
Set $\frac{\hat{\theta}}{\hat{\theta}+1} = \bar{Y}$, we get $\hat{\theta} = \frac{\bar{Y}}{1-\bar{Y}}$.
and $\theta_e = \frac{0.35}{1-0.35} = 0.54$.

Remarks

MM equations may have multiple solutions.

Exponential distribution. Both mean and variance are θ. In this case, take the lower order moments.

It may have no solutions, or the solutions may not be in the parameter space.

MM may not be applicable if there are not sufficient population moments.

Invariance property:

Let $\hat{\theta}_1, \cdots, \hat{\theta}_k$ be MME of $\theta_1, \cdots, \theta_k$, then the MME of $\tau(\theta) = \tau(\hat{\theta}_1, \cdots, \hat{\theta}_k)$

Let $X_1, \cdots, X_n$ iid $\sim$ Bernoulli (p). Find MME of $\tau(p) = \frac{1}{p}$.
solution: $\mu_1 = p, \hat{p} = \bar{X}_n$, so MME of $\tau(p) = \frac{1}{\hat{p}} = \frac{1}{\bar{X}_n}$.
$X_1, \cdots, X_n$ iid $\sim$ N (μ, σ^2).
Find MME of $\frac{\mu}{\sigma}$.
Solution: $\hat{\mu} = \bar{X}_n, \hat{\sigma}^2 = \frac{1}{n}\sum(X_i - \bar{X}_n)^2$.
MME of $\frac{\mu}{\sigma}$ is $\frac{\bar{X}_n}{\sqrt{\frac{1}{n}\sum(X_i - \bar{X}_n)^2}}$.

Binomial Model

Let $X_1, \cdots, X_n \sim$ iid Binomial (n, p) Estimate both n and p.
MM equations:
$\hat{n}\hat{p} = \bar{X}$
$\hat{n}\hat{p}(1-\hat{p}) + \hat{n}^2\hat{p}^2 = \frac{1}{n}\sum X_i^2$
$\hat{n} = \frac{\bar{X}^2}{\bar{X} - \frac{1}{n}\sum(X_i - \bar{X})^2}$
$\hat{p} = \frac{\bar{X}}{\hat{n}}$

Exercises

5.2.17. Find the MME of θ in the pdf
$f_Y(y;\theta) = (\theta^2 + \theta)y^{\theta-1}(1-y), 0 \leq y \leq 1.$
5.2.19. Find the MME for λ if a random sample of size n is taken from the exponential pdf $f_Y(y;\lambda) = \lambda e^{-\lambda y}, y \geq 0.$