Risk, return, and diversification

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OUTLINE

1. Introduction
2. Diversification and risk
3. Modern portfolio theory
4. Asset pricing models
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1. Introduction

As managers, we rarely consider investing in only one project at one time. Small businesses and large corporations alike can be viewed as a collection of different investments, made at different points in time. We refer to a collection of investments as a portfolio.

While we usually think of a portfolio as a collection of securities (stocks and bonds), we can also think of a business in much the same way -- a portfolios of assets such as buildings, inventories, trademarks, patents, et cetera. As managers, we are concerned about the overall risk of the company's portfolio of assets.

Suppose you invested in two assets, Thing One and Thing Two, having the following returns over the next year:

<table>
<thead>
<tr>
<th>Asset</th>
<th>Return</th>
</tr>
</thead>
<tbody>
<tr>
<td>Thing One</td>
<td>20%</td>
</tr>
<tr>
<td>Thing Two</td>
<td>8%</td>
</tr>
</tbody>
</table>

Suppose we invest equal amounts, say $10,000, in each asset for one year. At the end of the year we will have $10,000 (1 + 0.20) = $12,000 from Thing One and $10,000 (1 + 0.08) = $10,800 from Thing Two, or a total value of $22,800 from our original $20,000 investment. The return on our portfolio is therefore:

\[
\text{Return} = \left( \frac{22,800 - 20,000}{20,000} \right) = 14\%
\]

If instead, we invested $5,000 in Thing One and $15,000 in Thing Two, the value of our investment at the end of the year would be:

\[
\text{Value of investment} = 5,000 (1 + 0.20) + 15,000 (1 + 0.08) = 6,000 + 16,200 = 22,200
\]

and the return on our portfolio would be:

\[
\text{Return} = \left( \frac{22,200 - 20,000}{20,000} \right) = 11\%
\]

which we can also write as:
Return = \left[ \frac{\$5,000}{\$20,000} (0.2) \right] + \left[ \frac{\$15,000}{\$20,000} (0.08) \right] = 11\%

As you can see more immediately by the second calculation, the return on our portfolio is the \textit{weighted average} of the returns on the assets in the portfolio, where the weights are the proportion invested in each asset.

We can generalize the formula for a portfolio return, \( r_p \), as the weighted average of the returns of \textit{all} assets in the portfolio, letting:

- \( i \) indicate the particular asset in the portfolio,
- \( w_i \) indicate the proportion invested in asset \( i \),
- \( r_i \) indicate the return on asset \( i \), and
- \( S \) indicate the number of assets in the portfolio

The return on the portfolio is:

\[
r_p = w_1 r_1 + w_2 r_2 + \ldots + w_S r_S ,
\]

which we can write more compactly as:

\[
r_p = \sum_{i=1}^{S} w_i r_i
\]

**Example: The return on a portfolio**

**Problem**
Consider a portfolio comprised of three assets, with expected returns and investments of:

<table>
<thead>
<tr>
<th>Asset</th>
<th>Expected return</th>
<th>Investment</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>10%</td>
<td>$20,000</td>
</tr>
<tr>
<td>B</td>
<td>5%</td>
<td>$10,000</td>
</tr>
<tr>
<td>C</td>
<td>15%</td>
<td>$20,000</td>
</tr>
</tbody>
</table>

What is the expected return on the portfolio?

**Solution**
\[
\begin{align*}
    r_p &= 40\% \ (10\%) \quad + \quad 20\% \ (5\%) \quad + \quad 40\% \ (15\%) \\
    r_p &= 0.04 \quad + \quad 0.01 \quad + \quad 0.06 \\
    r_p &= 0.11 \text{ or } 11\%
\end{align*}
\]
2. Returns and the tolerance for bearing risk

Which product investment do you prefer, A or B? Most people would choose A since it provides the same expected return, with less risk. Most people do not like risk -- they are risk averse. Risk aversion is the dislike for risk. Does this mean a risk averse person will not take on risk? No -- they will take on risk if they feel they are compensated for it.

A risk neutral person is indifferent towards risk. Risk neutral persons do not need compensation for bearing risk. A risk preferent person likes risk -- someone even willing to pay to take on risk. Are there such people? Yes. Consider people who play the state lotteries, where the expected value is always negative: the expected value of the winnings is less than the cost of the lottery ticket.

When we consider financing and investment decisions, we assume that most people are risk averse. Managers, as agents for the owners, make decisions that consider risk "bad" and that if risk must be borne, they make sure there is sufficient compensation for bearing it. As agents for the owners, managers cannot have the "fun" of taking on risk for the pleasure of doing so.

Risk aversion is the link between return and risk. To evaluate a return you must consider its risk: Is there sufficient compensation (in the form of an expected return) for the investment's risk?

Example: Risk aversion and investor choices

Risk averse investors prefer more return to less, and prefer less risk to more.

Consider the following investments and the associated expected return and risk (measured by standard deviation):

<table>
<thead>
<tr>
<th>Investment</th>
<th>Expected return</th>
<th>Standard deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>10%</td>
<td>12%</td>
</tr>
<tr>
<td>B</td>
<td>10%</td>
<td>11%</td>
</tr>
<tr>
<td>C</td>
<td>11%</td>
<td>12%</td>
</tr>
<tr>
<td>D</td>
<td>11%</td>
<td>11%</td>
</tr>
<tr>
<td>E</td>
<td>9%</td>
<td>10%</td>
</tr>
<tr>
<td>F</td>
<td>12%</td>
<td>13%</td>
</tr>
</tbody>
</table>

If you are a risk-averse investor, which investment would you prefer of each of the following pairs:

- A or B?
- A or C?
- C or D?
- D or E?
- E or F?
- C or F?

Some choices are clear, and some are not. Some, like the choice between D and E, depend on the investor's individual preferences for risk and return tradeoff, which we refer to as their utility function.

3. Diversification and risk

"My ventures are not in one bottom trusted Nor to one place; nor is my whole estate Upon the fortune of this present year. Therefore my merchandise makes me not sad" - William Shakespeare, Merchant of Venice.

In any portfolio, one investment may do well while another does poorly. The projects' cash flows may be "out of synch" with one another. Let's see how this might happen.

Let's look at the idea of "out-of-synchness" in terms of expected returns, since this is what we face when we make financial decisions. Consider Investment One and Investment Two and their probability distributions:

<table>
<thead>
<tr>
<th>Scenario</th>
<th>Probability of Scenario</th>
<th>Return on Investment One</th>
<th>Return on Investment Two</th>
</tr>
</thead>
<tbody>
<tr>
<td>Boom</td>
<td>30%</td>
<td>20%</td>
<td>-10%</td>
</tr>
<tr>
<td>Normal</td>
<td>50%</td>
<td>0%</td>
<td>0%</td>
</tr>
</tbody>
</table>
We see that when Investment One does well, in the boom scenario, Investment Two does poorly. Also, when Investment One does poorly, as in the recession scenario, Investment Two does well. In other words, these investments are out of sync with one another.

Now let's look at how their "out-of-synchness" affects the risk of the portfolio of One and Two. If we invest an equal amount in One and Two, the portfolio's return under each scenario is the weighted average of One and Two's returns, where the weights are 50 percent:

<table>
<thead>
<tr>
<th>Scenario</th>
<th>Probability</th>
<th>Weighted average return</th>
</tr>
</thead>
<tbody>
<tr>
<td>Boom</td>
<td>0.30</td>
<td>[0.5 ( 0.20)] + [0.5 (-0.10)] = 0.0500 or 5%</td>
</tr>
<tr>
<td>Normal</td>
<td>0.50</td>
<td>[0.5 ( 0.00)] + [0.5 ( 0.00)] = 0.0000 or 10%</td>
</tr>
<tr>
<td>Recession</td>
<td>0.20</td>
<td>[0.5 (-0.20)] + [0.5 ( 0.45)] = 0.1250 or 12.5%</td>
</tr>
</tbody>
</table>

The calculation of the expected return and standard deviation for Investment One, Investment Two, and the portfolio consisting of One and Two results in the following statistics,

<table>
<thead>
<tr>
<th>Scenario</th>
<th>Probability of scenario</th>
<th>Return on Investment One</th>
<th>Return on Investment Two</th>
<th>Return on a portfolio comprised of One and Two</th>
</tr>
</thead>
<tbody>
<tr>
<td>Boom</td>
<td>30%</td>
<td>20%</td>
<td>-10%</td>
<td>5%</td>
</tr>
<tr>
<td>Normal</td>
<td>50%</td>
<td>0%</td>
<td>0%</td>
<td>0%</td>
</tr>
<tr>
<td>Recession</td>
<td>20%</td>
<td>-20%</td>
<td>45%</td>
<td>12.5%</td>
</tr>
<tr>
<td>Expected return</td>
<td>2%</td>
<td>6%</td>
<td>4%</td>
<td></td>
</tr>
<tr>
<td>Standard deviation</td>
<td>14.00%</td>
<td>19.97%</td>
<td>4.77%</td>
<td></td>
</tr>
</tbody>
</table>

The expected return on Investment One is 2 percent and the expected return on Investment Two is 6 percent. The return on a portfolio comprised of equal investments of One and Two is expected to be 4 percent. The standard deviation of Investment One's return is 14 percent and of Investment Two's return is 19.97 percent, but the portfolio's standard deviation, calculated using the weighted average of the returns on investments One and Two in each scenario, is 4.77 percent. This is less than the standard deviations of each of the individual investments because the returns of the two investments do not move in the same direction at the same time, but rather tend to move in opposite directions.

Try it: Expected value and Standard deviation

Consider the following distribution:

<table>
<thead>
<tr>
<th>Scenario</th>
<th>Probability</th>
<th>Outcome</th>
</tr>
</thead>
<tbody>
<tr>
<td>Boom</td>
<td>10%</td>
<td>50%</td>
</tr>
<tr>
<td>Normal</td>
<td>50%</td>
<td>10%</td>
</tr>
<tr>
<td>Bust</td>
<td>40%</td>
<td>-10%</td>
</tr>
</tbody>
</table>

What is the expected return and standard deviation of the distribution?

A. The role of covariance and correlation

The portfolio comprised of Investments One and Two has less risk than the individual investments because each moves in different directions with respect to the other. A statistical measure of how two variables -- in this case, the returns on two different investments -- move together is the **covariance**.
Covariance is a statistical measure of how one variable changes in relation to changes in another variable. Covariance in this example is calculated in four steps:

**Step 1:** For each scenario and investment, subtract the investment's expected value from its possible outcome;

**Step 2:** For each scenario, multiply the deviations for the two investments;

**Step 3:** Weight this product by the scenario's probability; and

**Step 4:** Sum these weighted products to arrive at the covariance.

<table>
<thead>
<tr>
<th>Scenario</th>
<th>Probability</th>
<th>Deviation of Investment One's return from its expected return</th>
<th>Deviation of Investment Two's return from its expected return</th>
<th>Product of the deviations</th>
<th>Weight the product by the probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>Boom</td>
<td>0.30</td>
<td>0.1800</td>
<td>-0.1600</td>
<td>-0.0288</td>
<td>-0.00864</td>
</tr>
<tr>
<td>Normal</td>
<td>0.50</td>
<td>-0.0200</td>
<td>-0.0600</td>
<td>0.0012</td>
<td>0.00060</td>
</tr>
<tr>
<td>Recession</td>
<td>0.20</td>
<td>-0.2200</td>
<td>0.3900</td>
<td>-0.0858</td>
<td>-0.01716</td>
</tr>
</tbody>
</table>

As you can see in these calculations, in a boom economic environment, when Investment One is above its expected return (deviation is positive), Investment Two is below its expected return (deviation is negative). In a recession, Investment One's return is below its expected value and Investment Two's return is above its expected value. The tendency is for the returns on these portfolios to co-vary in opposite directions -- producing a negative covariance of -0.0252.

We can represent this calculation in a formula, using \( p_i \) to represent the probability, \( r \) to represent the possible return, the \( E \) to represent the expected return:

\[
\text{Covariance}_{\text{One,Two}} = \sum_{i=1}^{N} p_i (r_{\text{One},i} - E_{\text{One}})(r_{\text{Two},i} - E_{\text{Two}})
\]

Let's see the effect of this negative covariance on the risk of the portfolio. The portfolio's variance depends on:

1. the weight of each asset in the portfolio;
2. the standard deviation of each asset in the portfolio; and
3. the covariance of the assets' returns.

Let \( \text{cov}_{1,2} \) represent the covariance of two assets' returns. We can write the portfolio variance for a two-security portfolio as:

\[
\text{Portfolio variance, 2-security portfolio} = w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2 + 2 w_1 w_2 \text{cov}_{1,2}.
\]

The portfolio standard deviation is the square root of the variance, or:

\[
\text{Portfolio standard deviation, 2-security portfolio} = \sqrt{w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2 + 2 w_1 w_2 \text{cov}_{1,2}}.
\]

More generally, the formula is:

\[
\text{Portfolio variance, } n\text{-security portfolio} = \sum_{i=1}^{n} w_i^2 \sigma_i^2 + \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} 2 w_i w_j \text{cov}_{ij}.
\]

\[
\text{Portfolio standard deviation, } n\text{-security portfolio} = \sqrt{\sum_{i=1}^{n} w_i^2 \sigma_i^2 + \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} 2 w_i w_j \text{cov}_{ij}}.
\]

---

1 You should notice a similarity between the calculation of the covariance and the variance (that you learned in the Measuring Risk reading). In the case of the variance, we took the deviation from the expected value and squared it before weighting it by the probability. In the case of the covariance, we take the deviations for each asset, multiply them, and then weight by the probability.
Recognizing that the covariance is the product of the correlation and the respective standard deviations, we can also write the formula as:

\[
\text{Portfolio standard deviation} = \sqrt{\sum_{i=1}^{N} w_i^2 \sigma_i^2 + \sum_{i=1}^{N} \sum_{j \neq i} w_i w_j \sigma_{ij} \sigma_{ij}}.
\]

We can apply this general formula to our example, with Investment One's characteristics indicated with a 1 and Investment Two's with a 2,

- \(w_1 = 0.50\) or 50 percent
- \(w_2 = 0.50\) or 50 percent
- \(\sigma_1 = 0.1400\) or 14.00 percent
- \(\sigma_2 = 0.1997\) or 19.97 percent
- \(\text{cov}_{1,2} = -0.0252\).

The portfolio variance, \(\sigma_p^2\), is:

\[
\sigma_p^2 = 0.50^2(0.1400^2) + 0.50^2(0.1997^2) + 2(0.50)(0.50)(-0.0252) = 0.002275,
\]

and the portfolio standard deviation, \(\sigma_p\), is 0.0477 or 4.77 percent, which, not coincidentally, is what we got when we calculated the standard deviation directly from the portfolio returns under the three scenarios.

The standard deviation of the portfolio is lower than the standard deviations of each of the investments because the returns on Investments One and Two are **negatively related**: when one is doing well the other may be doing poorly, and vice-versa. That is, the covariance is negative.

### Example: The portfolio variance and standard deviation

**Problem**

Consider a portfolio comprised of two securities, F and G:

<table>
<thead>
<tr>
<th>Statistic</th>
<th>Security F</th>
<th>Security G</th>
</tr>
</thead>
<tbody>
<tr>
<td>Expected return</td>
<td>10%</td>
<td>20%</td>
</tr>
<tr>
<td>Standard deviation</td>
<td>5%</td>
<td>8%</td>
</tr>
<tr>
<td>Percentage of portfolio invested</td>
<td>40%</td>
<td>60%</td>
</tr>
</tbody>
</table>

The covariance between the two securities' returns is 0.002. What is the portfolio's standard deviation?

**Solution**

\[
\begin{align*}
\sigma_{p}^2 &= 0.16 (0.0025) + 0.36 (0.0064) + [2(0.002)(0.040)(0.60)] \\
\sigma_{p}^2 &= 0.0004 + 0.0023 + 0.00096 = 0.00366 \\
\sigma_{p} &= \textbf{0.06050} \text{ or 6.05 percent}
\end{align*}
\]

The returns on two investments are:

---

2 Correlation is a statistical measure of association.
• **Positively correlated** if one's returns tend to vary in the same direction at the same time as the other's returns.

• **Negatively correlated** if one's returns tend to vary in the opposite direction with respect to the other's returns.

• **Uncorrelated** if there is no relation between the changes in one's returns with changes in the other's returns.

Statistically, we can measure correlation with a correlation coefficient $\rho$. The correlation coefficient reflects how the returns of two securities vary together and is measured by the covariance of the two securities' returns, divided by the product of their standard deviations:

$$
\text{Correlation coefficient} = \frac{\text{covariance of the two assets' returns}}{\left( \text{standard deviation of returns on first asset} \right) \left( \text{standard deviation of returns on second asset} \right)}
$$

$$
\rho_{1,2} = \frac{\text{cov}_{1,2}}{\sigma_1 \sigma_2}
$$

By construction, the correlation coefficient is bound between -1 and +1. We can interpret the correlation coefficient as follows:

If the correlation coefficient is ...
- +1  a perfect, positive correlation between the two assets' returns.
- -1  a perfect, negative correlation between the two assets returns.
- 0  no correlation between the two assets returns.
- between 0 and +1 positive, but not perfect positive correlation between the two assets returns, as illustrated in Exhibit 1.
- between -1 and 0 negative, but not perfect negative correlation between the two assets returns, as illustrated in Exhibit 2.

In the case of Investments One and Two, the covariance of their returns is between -1 and 0. Therefore, we say that the returns on Investment One and Investment Two are negatively correlated with one another.

By investing in assets with less than perfectly correlated cash flows, you are getting rid of -- diversifying away -- some risk. The less correlated the cash flows, the more risk you can diversify away -- to a point.
Let's see how the correlation and portfolio standard deviation interact. Consider two investments, E and F, whose standard deviations are 5 percent and 3 percent, respectively. Suppose our portfolio consists of an equal investment in each; that is, \( w_1 = w_2 = 50 \) percent.

<table>
<thead>
<tr>
<th>If the correlation between the assets' returns is ...</th>
<th>... this means that the covariance is ...</th>
<th>and the portfolio's standard deviation is ...</th>
</tr>
</thead>
<tbody>
<tr>
<td>+1.0</td>
<td>+0.0150</td>
<td>4.00%</td>
</tr>
<tr>
<td>+0.5</td>
<td>+0.0075</td>
<td>3.50%</td>
</tr>
<tr>
<td>0.0</td>
<td>0.0000</td>
<td>2.92%</td>
</tr>
<tr>
<td>-0.5</td>
<td>-0.0075</td>
<td>2.18%</td>
</tr>
<tr>
<td>-1.0</td>
<td>-0.0150</td>
<td>0.00%</td>
</tr>
</tbody>
</table>

The less perfectly positively correlated are two assets' returns, the lower the risk of the portfolio comprised of these assets.

### Example: Correlation and covariance

**Problem**

Consider Securities F and G:

- Standard deviation of F's returns = 5%
- Standard deviation of G's returns = 8%
- Covariance of F & G's returns = 0.002

What is the correlation between F & G's returns?

**Solution**

\[
\text{Correlation} = \frac{0.002}{(0.05)(0.08)} = 0.50
\]

### Try it! Portfolio risk I

Suppose you are given the following information on two stocks, Tweedle Dee and Tweedle Dum:

<table>
<thead>
<tr>
<th></th>
<th>Tweedle Dee</th>
<th>Tweedle Dum</th>
</tr>
</thead>
<tbody>
<tr>
<td>Expected return</td>
<td>6%</td>
<td>10%</td>
</tr>
<tr>
<td>Standard deviation of returns</td>
<td>8%</td>
<td>12%</td>
</tr>
</tbody>
</table>

The correlation of the returns for Tweedle Dee and Tweedle Dum is 0.35

If you invest 40 percent of your portfolio in Tweedle Dee and the remainder in Tweedle Dum, what is the expected return and standard deviation for your portfolio?

Let's think about what this means for a company. Consider Proctor & Gamble whose products include Tide detergent, Prell shampoo, Pampers diapers, Jif peanut butter, and Old Spice cologne. Are the cash flows from these products positively correlated? To a degree, yes. The cash flows from these products depend on consumer spending for consumption goods. But are they perfectly correlated? No. For example, diaper sales depend on the diaper wearing population, whereas cologne products depend on the male cologne-wearing population. The cash flows of these different products also depend on the actions of competitors -- the degree of competition may be different for the diaper market than the peanut butter market. Further, the cash flows of the products are affected by different input pricing - the costs of the raw inputs to make these products. If there is a bad year for the peanut crop, the price of
peanuts may increase substantially, reducing cash flows from Jif -- but this increase in peanut prices is not likely to affect the costs of, say, producing laundry detergent.

The effects of diversification on portfolio risk

<table>
<thead>
<tr>
<th>Returns on individual stocks, A and B</th>
<th>Returns on a portfolio of stocks comprised of equal parts of A and B</th>
</tr>
</thead>
<tbody>
<tr>
<td>Correlation between the returns on A and B = -0.71</td>
<td></td>
</tr>
</tbody>
</table>

![Graph showing returns on individual stocks A and B and a portfolio of A and B]

Try it! Portfolio risk II

Consider a portfolio comprised of 40% Stock A and 60% Stock B with the following characteristics:

<table>
<thead>
<tr>
<th>Stock</th>
<th>Expected return</th>
<th>Standard deviation</th>
<th>Weight</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>10%</td>
<td>5%</td>
<td>40%</td>
</tr>
<tr>
<td>B</td>
<td>20%</td>
<td>25%</td>
<td>60%</td>
</tr>
</tbody>
</table>

Correlation 0.2

- What is the expected portfolio return?
- What is the portfolio standard deviation?

B. Portfolio size and risk

What we have seen for a portfolio with two assets can be extended to include any number of assets. Alas, the calculations get more complicated because we have to consider the covariance between every possible pair of assets. But the basic idea is the same. The risk of a portfolio declines as it is expanded.
with assets whose returns are not perfectly correlated with the returns of the assets already in the portfolio.

The idea of diversification is based on beliefs about what will happen in the future: expected returns, standard deviation of all possible returns, and expected covariance between returns. How valid are our beliefs about anything in the future? We can get an idea by looking at the past. So we look at historical returns on assets -- returns over time -- to get an idea of how some asset's returns increase while at the same time others do not or decline.

Let's look at the effects of diversification with common stocks. As we add common stocks to a portfolio, the standard deviation of returns on the portfolio declines -- to a point. This is illustrated in Exhibit 3, where the ratio of the amount of risk is shown on the vertical axis as a ratio of the portfolio's standard deviation to the typical individual security's standard deviation. After around twenty different stocks, the portfolio's standard deviation is about as low as it is going to get. Why does the risk seem to reach some point and not decline any farther? Because common stocks' returns are generally positively correlated with one another. There just aren't enough negatively correlated stocks' returns to reduce portfolio risk beyond a certain point.

We refer the risk that goes away as we add assets as **diversifiable risk**. We refer to the risk that cannot be reduced by adding more assets as **non-diversifiable risk**.

Risk is not diversified only with common stocks. Consider a business that has many different product lines. While the returns on these product lines may not be perfectly correlated, the returns are influenced by similar forces -- for example, the economy and industry competition -- so that the returns are most likely positively correlated.

### Exhibit 3  Diversification illustration

As more assets are added to the portfolio, the portfolio's risk declines, eventually leveling off.

![Diversification illustration](image)

4. **Modern portfolio theory**

The idea that we can reduce the risk of a portfolio by introducing assets whose returns are not highly correlated with one another is the basis of **Modern portfolio theory (MPT)**. MPT tells us that by combining assets whose returns are not correlated with one another, we can determine combinations of assets that provide the least risk for each possible expected portfolio return.

Though the mathematics involved in determining the optimal combinations of assets are beyond this text, the basic idea is provided in Exhibit 4.
Each blue colored point in Panel A in Exhibit 4 represents a possible portfolio that can be put together comprising different assets and different weights. The points in this graph represent every possible portfolio. As you can see in this diagram:

- some portfolios have a higher expected return than other portfolios with the same level of risk;
- some portfolios have a lower standard deviation than other portfolios with the same expected return.

Investors are risk averse: they more return to less return and prefer less risk to more risk. Therefore, some portfolios are better than others. The best portfolios -- those that can't be beat in terms of either the level of return for the amount of risk or the amount of risk for the level of return -- make up what is called the efficient frontier. If investors are rational, they will go for the portfolios that fall on this efficient frontier. All the possible portfolios and the efficient frontier (shown in green) are diagrammed in Panel B of Exhibit 4.

So what portfolio would an investor choose? A rational investor would choose a portfolio that has a better risk-return combination than others; therefore, a rational investor would choose a portfolio that falls along the efficient frontier. But which of these many portfolios along the frontier would an investor select? An investor would have to address the issue of their tolerance for risk and would select the portfolio along this frontier that is the level of risk that is preferred.³

What is the relevance of MPT to financial managers? MPT tells us that

- we can manage risk by judicious combinations of assets in our portfolios; and
- there are some combinations of assets that are preferred over others.

³ In economics, we referred to this tradeoff between risk and return is the investor's utility function.
The relation between portfolio returns and portfolio risk was recognized by two Nobel Laureates in Economics, Harry Markowitz and William Sharpe. Harry Markowitz tuned us into the idea that investors hold portfolios of assets and therefore our focus is upon the portfolio return and the portfolio risk, not on the return and risk of individual assets.\(^4\) Is this reasonable? Probably. Not many businesses consists of a single asset. Nor do investors invest in only one asset.

The relevant risk to an investor is the portfolio's risk, not the risk of an individual asset. If an investor holds assets in a portfolio and is considering buying an additional asset or selling an asset from the portfolio, what must be considered is how this change will affect the risk of the portfolio. This concept applies whether we are talking about an investor holding 30 different stocks or a business that has invested in 30 different projects. The important thing in valuing an asset is its contribution to the portfolio's return and risk.

5. Asset pricing models

A. The capital asset pricing model

William Sharpe took the idea that portfolio return and risk are the only elements to consider and developed a model that deals with how assets are priced. This model is referred to as the capital asset pricing model (CAPM).

We just saw that there is a set of portfolios that make up the efficient frontier -- the best combinations of expected return and standard deviation. All the assets in each portfolio, even on the frontier, have some risk. Now let's see what happens when we add an asset with no risk -- referred to as the risk-free asset. Suppose we have a portfolio along the efficient frontier that has a return of 4 percent and a standard deviation of 3 percent. Suppose we introduce into this portfolio the risk-free asset, which has an expected return of 2 percent and, by definition, a standard deviation of zero. If the risk-free asset's expected return is certain, there is no covariance between the risky portfolio's returns and the returns of the risk-free asset.

A portfolio comprised of 50 percent of the risky portfolio and 50 percent of the risk-free asset has an expected return of \((0.50) 4\% + (0.50) 2\% = 3\%\) and a portfolio standard deviation calculated as follows:

\[
\text{Portfolio variance} = \sigma_p^2 = 0.50^2(0.03) + 0.50^2(0.00) + 2 (0.00) 0.50 (0.50) = 0.0075.
\]

\[
\text{Portfolio standard deviation} = \sigma_p = 0.0866.
\]

If we look at all possible combinations of portfolios along the efficient frontier and the risk-free asset, we see that the best portfolios are no longer along the entire length of the efficient frontier, but rather are the combinations of the risk-free asset and one -- and only one -- portfolio of risky assets on the frontier. The combinations of the risk-free asset and this one portfolio is shown in Exhibit 4. These combinations differ from one another by the proportion invested in the risk-free asset; as less is invested the risk-free asset, both the portfolio's expected return and standard deviation increase.

William Sharpe demonstrates that this one and only one portfolio of risky assets is the market portfolio -- a portfolio that consists of all assets, with the weights of these assets being the ratio of their market value to the total market value of all assets.

If investors are all risk averse -- they only take on risk if there is adequate compensation -- and if they are free to invest in the risky assets as well as the risk-free asset, the best deals lie along the line that is tangent to the efficient frontier. This line is referred to as the capital market line (CML), shown in Exhibit 5.

If the portfolios along the capital market line are the best deals and are available to all investors, it follows that the returns of these risky assets will be priced to compensate investors for the risk they bear relative to that of the market portfolio. Since the portfolios along the capital market line are the best deals, they are as diversified as they can get -- no other combination of risky assets or risk-free asset provides a better expected return for the level of risk or provides a lower risk for the level of expected return.

The capital market line tells us about the returns an investor can expect for a given level of risk. The CAPM uses this relationship between expected return and risk to describe how assets are priced. The CAPM specifies that the return on any asset is a function of the return on a risk-free asset plus a risk premium. The return on the risk-free asset is compensation for the time value of money. The risk premium is the compensation for bearing risk. Putting these components of return together, the CAPM says:

\[
\text{Expected return on an asset} = \text{expected return on a risk free asset} + \text{risk premium}
\]

In other words, the expected return on an asset is the sum of the compensation for the time value of money and compensation for bearing risk.

The market portfolio therefore represents the most well-diversified portfolio -- the only risk in a portfolio comprising all assets is the non-diversifiable risk. As far as diversification goes, the market portfolio is the best you can do, since you have included everything in it.

By the same token, if we assume that investors hold well-diversified portfolios (approximating the market portfolio), the only risk they have is non-diversifiable risk. If assets are priced to compensate for the risk of assets and if the only risk in your portfolio is non-diversifiable risk, then it follows that the compensation for risk is only for non-diversifiable risk. Let's refer to this non-diversifiable risk as market risk.

Since the market portfolio is made up of all assets, each asset possesses some degree of market risk. Since market risk is systematic across assets, it is often referred to as systematic risk and diversifiable risk is referred to as unsystematic risk. Further, the risk that is not associated with the market as a whole is often referred to as company-specific risk when referring to stocks, since it is risk that is specific to the company's own situation -- such as the risk of lawsuits and labor strikes -- and is not part of the risk that pervades all securities.

The measure of an asset's return sensitivity to the market's return, its market risk, is referred to as that asset's beta.

The expected return on an individual asset is the sum of the expected return on the risk-free asset and the premium for bearing

---

5 We often use major indices, such as the S&P 500, as proxies for the market portfolio.
market risk. Let \( r_i \) represent the expected return on asset \( i \), \( r_f \) represented the expected return on the risk-free asset, and \( b_i \) represent the degree of market risk for asset \( i \). Then:

\[
    r_i = r_f + (r_m - r_f) \beta_i
\]

The term \( (r_m - r_f) \), is the **market risk premium** -- if you owned all the assets in the market portfolio, you expect to be compensated \( (r_m - r_f) \) for bearing the risk of these assets. \( b \) is measure of market risk, which serves to fine-tune the risk premium for the individual asset. For example, if the market risk premium were 2 percent and the \( b \) for an individual asset were 1.5, you would expect to receive a risk premium of 3 percent since you are taking on 50 percent more risk than the market.

For each asset there is a beta. If we represent the expected return on each asset and its beta as a point on a graph, and we do the same for every asset in the market, and connect all the points, the result is the **security market line**, SML, as shown in Exhibit 6.

Exhibit 6  The security market line

For an individual asset, \( b \) is a measure of sensitivity of its returns to changes in return on the market portfolio.

- If \( b \) is **one**, we expect that for a given change of 1 percent in the market portfolio return, the asset's return is expected to change by 1 percent.
- If \( b \) is **less than one**, then for a 1 percent change in the expected market return, the asset's return is expected to change by less than 1 percent.
- If \( b \) is **greater than one**, then for a 1 percent change in the expected market return, the asset's return is expected to change by more than 1 percent.

---

6 In economics class, you referred to this sensitivity as **elasticity**.
We typically estimate the $\beta$ for a common stock by looking at the historical relation between its return and the return on the market as a whole. The $\beta$s of some firms' stocks are close to one, indicating that the returns on these stocks tend to move along with the market. There are some betas around 0.3 (Homestake Mining and Southern Company, for example), indicating that the returns on these securities do not move along with the market: if the market were to go up 10 percent, we would expect these securities' returns to go up only around 3 percent. Then there are some stocks whose beta is much higher than 1.0. For example, if J-Mart has a $\beta$ of 1.5. This means that if the market is expected to go up 1 percent, we expect J-Mart's return to go up 1.5 percent; if the market is expected to go down 1 percent, we expect J-Mart's return to go down 1.5 percent.

When we introduced the idea of risk, we discussed many different types of risk, many of which affect cash flows. For example, every firm has some type of business risk. And some of that business risk is common among all firms -- all firm's sales are affected by the economy to some extent. But some of that business risk an investor can diversify away by buying stocks whose sensitivity to the economy are out of synch with one another. However, part of business risk cannot be diversified away -- we are stuck with it. This is the risk that concerns investors and they want to be compensated for it.

If we know part of the risk of a particular asset is common to all assets, and we have a large enough representation of all in the assets in our portfolio, then we don't need to be concerned with the diversifiable risk. We are concerned about the market risk of each asset in the portfolio and how it contributes to the market risk of the entire portfolio.

We can get a good idea of the portfolio's market risk by using a $\beta$ that represents the composition of the assets in the portfolio. To determine the portfolio's beta, $\beta_p$, we need to know the weighted average of the betas of the assets that make up the portfolio, where each weight is the proportion invested in each asset. Let $\beta_i$ indicate the beta of the portfolio, $w_i$ indicate the proportion invested in each the asset $i$, and $\beta_i$ the beta for asset $i$. If there are $S$ assets in the portfolio, then:

$$\beta_p = w_1 \beta_1 + w_2 \beta_2 + w_3 \beta_3 + \ldots + w_S \beta_S,$$

or more compactly,

$$\beta_p = \sum_{i=1}^{S} w_i \beta_i$$

Suppose we have three securities in our portfolio, with the amount invested in each and their security beta as follows:

<table>
<thead>
<tr>
<th>Security</th>
<th>Security beta</th>
<th>Amount invested</th>
</tr>
</thead>
<tbody>
<tr>
<td>AAA</td>
<td>1.00</td>
<td>$10,000</td>
</tr>
<tr>
<td>BBB</td>
<td>1.50</td>
<td>$20,000</td>
</tr>
<tr>
<td>CCC</td>
<td>0.75</td>
<td>$20,000</td>
</tr>
</tbody>
</table>

The portfolio's beta is:

\[
\beta_p = \sum_{i=1}^{S} w_i \beta_i
\]
If the expected risk-free rate of interest is 4 percent and the expected return on the market is 7 percent, the $\beta_p = 1.1$ means that the expected return on portfolio is 4 percent + 1.10 (7 percent - 4 percent) = 7.3 percent.

The CAPM, with its description of the relation between expected return and risk and the importance of market risk in asset pricing, has some drawbacks.

First, the estimate of $\beta$ is just that: an estimate. For stocks, the $\beta$ is typically estimated using historical returns. But the proxy for market risk depends on the method and period in which it is measured. For assets other than stocks, $\beta$ estimation is more difficult.

Second, the CAPM includes some unrealistic assumptions. For example, it assumes that all investors can borrow and lend at the same rate. Though the conclusions of the CAPM are logical, in terms of the role of diversification and the market portfolio, the theoretical model is based on a large number of assumptions, some of which are unrealistic (e.g., all investors have the same expectations regarding risk and return for all assets).

Third, the CAPM is a theory that cannot be tested. The market portfolio is a theoretical construct (that is, all investable assets) and not really observable, so we cannot test the relation between the expected return on an asset and the expected return of the market to see if the relation specified in the CAPM holds.

Lastly, in studies of the CAPM applied to common stocks, the CAPM does not explain the differences in returns for securities that differ over time, differ on the basis of dividend yield, and differ on the basis of the market value of equity (the so called "size effect").

Though it lacks realism and is difficult to apply, the CAPM makes some sense regarding the role of diversification and the type of risk we need to consider in investment decisions.

---

Example: The Portfolio Beta

**Problem**
Consider a portfolio comprised of three securities with the following security betas and proportions invested in each:

<table>
<thead>
<tr>
<th>Security</th>
<th>Security beta</th>
<th>Proportion invested</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.00</td>
<td>50%</td>
</tr>
<tr>
<td>2</td>
<td>2.00</td>
<td>25%</td>
</tr>
<tr>
<td>3</td>
<td>0.50</td>
<td>25%</td>
</tr>
</tbody>
</table>

**Solution:**
Portfolio beta = 50% (1.00) + 25% (2.00) + 25% (0.50)
Portfolio beta = 0.50 + 0.50 + 0.125 = 1.125

---

Try it! Expected return using the CAPM

For the stocks in Exhibit 7, calculate the expected return on each stock if the expected risk-free rate of return is 3 percent and the expected risk premium on the market is 4 percent.

---

B. The arbitrage pricing model

An alternative to CAPM in relating risk and return is the Arbitrage Pricing Model. The *arbitrage pricing model* developed by Stephen Ross, is an asset pricing model that is based on the idea that identical assets in different markets should be priced identically.

While the CAPM is based on a market portfolio of assets, the arbitrage pricing model doesn't mention a market portfolio at all. Instead, the arbitrage pricing model states that an asset's returns should compensate the investor for the risk of the asset, where the risk is due to a number of economic influences, or economic factors. Therefore, the expected return on the asset, $r_i$, is:

$$r_i = r_f + \delta_1 f_1 + \delta_2 f_2 + \delta_3 f_3 + \ldots$$

where each of the $\delta$ 's reflect the asset's return's sensitivity to the corresponding economic factor, $f$. The arbitrage pricing model looks much like the CAPM, but the CAPM has one factor -- the market portfolio. There are many factors in the Arbitrage Pricing Model.

What if an asset's price is such that it is out of line with what is expected? That's where arbitrage comes in. Any time an asset's price is out of line with how market participants feel it should be priced -- based on the basic economic influences -- investors will enter the market and buy or sell the asset until its price is in line with what investors think it should be.

What are these economic factors? They are not specified in the original Arbitrage Pricing Model, though evidence suggests that these factors include:

- Unanticipated changes in inflation;
- Unanticipated changes in industrial production;
- Unanticipated changes in risk premiums; and
- Unanticipated changes in the difference between interest rates for short and long term securities.

Anticipated factors are already reflected in an asset's price. It is the *un*anticipated factors that cause an asset's price to change. For example, consider a bond with a fixed coupon interest. The bond's current price is the present value of expected interest and principal payments, discounted at some rate that reflects the time value of money, the uncertainty of these future cash flows, and the expected rate of inflation. If there is an unanticipated increase in inflation, what will happen to the price of the bond? It will go down since the discount rate increases as inflation increases. And if the price of the bond goes down, so too does the return on the bond. Therefore, the sensitivity of a bond's price to changes in unanticipated inflation is negative.

The Arbitrage Pricing Model is not without drawbacks. First, the factor sensitivities must be estimated. The model is based on the sensitivities of expected returns to unanticipated changes in the factors. Alas, the best we can do, much like the CAPM, is look at historical relationships. Second, some financial observers argue that a single factor, namely the market portfolio, does just as good a job in explaining security returns as the more complex multiple factor approach of the Arbitrage Pricing Model.

C. Financial decision-making and asset pricing

Portfolio theory and asset pricing models lay the groundwork for financial decisions. While portfolio theory and asset pricing theory are complex and rely on many assumptions, they do get us thinking about what is important:

- Return and risk must both be considered.
- Since investors must be compensated for risk, a greater return is expected for bearing greater risk.
• Investors hold portfolios of assets; therefore the relevant risk in the valuation of assets is the portfolio's risk.

If a corporation is considering investing in a new product, there are two levels of thinking to work through in evaluating its risk and returns:

• If a firm takes on the product, it is adding it to its portfolio of assets and needs to consider the affect of this product on the firm's overall risk.
• Since a firm is owned by investors, who themselves may own portfolios of assets, the relevant risk to consider is how the change in the firm's risk affects the owners' portfolio risk.

Therefore, when we evaluate the new product's future cash flows, the discount rate that we apply to value these future cash flows must reflect how that product affects the owners' portfolio risk.

6. Summary

Portfolio theory and the related mathematics help us understand the relation between risk and return. Though portfolio theory is often demonstrated in terms of investing in stocks, the concepts are much more comprehensive. Consider that every business is a portfolio of assets, some tangible and some intangible. By understanding how diversification works, a financial manager gains a better understanding of the relevant risks in decision-making and, hence, a better understanding of valuation of investments.
7. Solutions to **Try it!**

**Expected value and Standard deviation**

Expected return = 6%
Standard deviation = 17.443%

**Portfolio risk I**

Standard deviation = \(\sqrt{0.00102 + 0.00518 + 2(0.000806)} = 8.843\%\)

**Portfolio risk II**

Expected return = 16%
Portfolio standard deviation = 15.52%

**Expected return using the CAPM**

<table>
<thead>
<tr>
<th>Common stock</th>
<th>(\beta)</th>
<th>(r_i)</th>
</tr>
</thead>
<tbody>
<tr>
<td>3Com</td>
<td>1.15</td>
<td>7.6%</td>
</tr>
<tr>
<td>Advance Auto Parts</td>
<td>0.90</td>
<td>6.6%</td>
</tr>
<tr>
<td>Archer-Daniels Midland</td>
<td>0.70</td>
<td>5.8%</td>
</tr>
<tr>
<td>Big Lots</td>
<td>1.10</td>
<td>7.4%</td>
</tr>
<tr>
<td>BJ’s Wholesale</td>
<td>1.05</td>
<td>7.2%</td>
</tr>
<tr>
<td>Claire’s Stores</td>
<td>1.10</td>
<td>7.4%</td>
</tr>
<tr>
<td>PetSmart</td>
<td>1.00</td>
<td>7.0%</td>
</tr>
<tr>
<td>Reynolds American</td>
<td>0.95</td>
<td>6.8%</td>
</tr>
<tr>
<td>Seagate Technologies</td>
<td>1.20</td>
<td>7.8%</td>
</tr>
<tr>
<td>Stillwater Mining</td>
<td>1.30</td>
<td>8.2%</td>
</tr>
<tr>
<td>W.M. Wrigley</td>
<td>0.55</td>
<td>5.2%</td>
</tr>
<tr>
<td>Williams-Sonoma</td>
<td>1.25</td>
<td>8.0%</td>
</tr>
</tbody>
</table>

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