Contents
Step 1: Calculate the spot rates corresponding to the yields
Step 2: Calculate the one-year forward rates for each relevant year ahead
Step 3: Diagram the lattice
Step 4: Calculate the forward rates considering interest rate volatility
Step 5: Calculate the value of the security by discounting through the lattice
We use interest rate lattices, also known as interest rate trees, to demonstrate a number of valuation issues related to fixed income securities. For example, we can examine the effect of interest rate volatility on the price of bonds. This is especially useful in valuing a security with an embedded option, such as a callable bond.

There is not just one method used to construct a lattice because a lattice depends on the type of valuation and the assumptions regarding interest rate volatility. However, once of the basics of the interest rate lattice are laid out, modifications for specific applications is straightforward.

The example that we will use is the following:

What is the value of an annual-pay corporate bond that has a coupon rate of 6 percent, three years to maturity? If the relevant yields for securities trading at par:

<table>
<thead>
<tr>
<th>Maturity, in years</th>
<th>Yields</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4%</td>
</tr>
<tr>
<td>2</td>
<td>5%</td>
</tr>
<tr>
<td>3</td>
<td>6%</td>
</tr>
</tbody>
</table>

The yields are what we observe for a one-year, two-year, and three-year security respectively. If we value the bond based on the yield to maturity, the 6 percent bond has a value of $100. However, these yields each assume that the yield curve is flat; for example, the two-year yield assumes that the cash flow occurring in the first year is reinvested at the two-year yield.

If we want to allow for a non-flat yield curve, we must calculate the rates for the cash flows occurring at the different time periods, such that the first year interest on both the two-year and the three-year securities are discounted at the same rate, the second year cash flow for both the two-year and the three year securities are discounted at the same rate, and so on. We refer to these rates as the spot rates because these rates reflect the interest rates implied by the current yields.

**Step 1: Calculate the spot rates corresponding to the yields**

The spot rate for the first year is, simply, the yield for the one-year security. It’s calculating the spot rates for the periods that follow it where it gets messy.

We can use bootstrapping to determine the forward rates from the spot rates for the second and third periods. Bootstrapping is, simply, sequential calculations: once we determine the spot rate for the second period, we can then determine the spot rate for the third period. For this set of spot rates, the calculations are the following:

\[
100 = \frac{5}{(1+0.04)^1} + \frac{105}{(1+x)^2}
\]

\[
100 = \frac{6}{(1+0.04)^1} + \frac{6}{(1+x)^1} + \frac{106}{(1+y)^2}
\]

We are using \(x\) for the two-year spot rate and \(y\) for the three-year spot rate.
In the bootstrap process, we first solve for $x$, and then insert $x$ into the second equation and solve for $y$:

\[
100 = \frac{5}{(1+0.04)^1} + \frac{105}{(1+x)^2}
\]

The basic relationship

\[
100 = 4.80769 + \frac{105}{(1+x)^2}
\]

Calculate the value for present value of the first period’s cash flow

\[
95.19231 = \frac{105}{(1+x)^2}
\]

Subtract the present value of the first period’s cash flow from the value of the security.

\[
\frac{95.19231}{105} = \frac{1}{(1+x)^2}
\]

Divide both sides by the second period’s cash flow.

\[
\frac{105}{95.19231} = (1+x)^2
\]

Invert both sides.

\[
\sqrt[2]{\frac{105}{95.19231}} = (1+x)
\]

Take the square root of both sides.

\[
1.050252 = (1+x)
\]

Calculate the left-hand side.

\[
x = 0.050252
\]

Subtract one from both sides.

Now that we have the spot rate for the two-year security, we now calculate the rate for the three-year security:

\[
100 = \frac{6}{(1+0.04)^1} + \frac{6}{(1+0.050252)^2} + \frac{106}{(1+y)^3}
\]

The basic relationship

\[
100 = 5.76923 + 5.439566 + \frac{106}{(1+y)^3}
\]

Calculate the value for present value of the first period’s cash flow

\[
88.791204 = \frac{106}{(1+y)^3}
\]

Subtract the present value of the first and second periods’ cash flow from the value of the security.

\[
\frac{88.791204}{106} = \frac{1}{(1+y)^3}
\]

Divide both sides by the third period’s cash flow.

\[
\frac{106}{88.791204} = (1+y)^3
\]

Invert both sides.

\[
\sqrt[3]{\frac{106}{88.791204}} = (1+y)
\]

Take the third root of both sides.

\[
1.060829 = (1+y)
\]

Calculate the left-hand side.

\[
y = 0.060829
\]

Subtract one from both sides.

We now have the following information:
<table>
<thead>
<tr>
<th>Maturity, in years</th>
<th>Yields</th>
<th>Spot rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4%</td>
<td>4%</td>
</tr>
<tr>
<td>2</td>
<td>5%</td>
<td>5.0252%</td>
</tr>
<tr>
<td>3</td>
<td>6%</td>
<td>6.0829%</td>
</tr>
</tbody>
</table>

If we value the bond based on the spot rates, we have a bond value of $100.00:

Cash flow | Spot rate | Discounted cash flow |
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$6.00</td>
<td>4.0000%</td>
<td>$5.769</td>
</tr>
<tr>
<td>$6.00</td>
<td>5.0252%</td>
<td>5.440</td>
</tr>
<tr>
<td>$106.00</td>
<td>6.0829%</td>
<td>88.791</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$100.00</td>
</tr>
</tbody>
</table>

**Step 2: Calculate the one-year forward rates for each relevant year ahead**

Once we calculate the spot rates, we can now calculate the one-year forward rates. A forward rate is a rate anticipated in the future for a specific maturity at a specific point in time in the future. In this analysis, we would like to calculate the one-year rate one year from now and the one-year rate two years from now.

The relation between the spot rates and the forward rates are the following:

\[ f_n = \left( \frac{1+s_{n+1}}{1+s_n} \right)^{n+1} - 1 \]

where

- \( f_n \) is the forward rate \( n \) periods from today;
- \( s_n \) is the spot rate for period \( n \); and
- \( s_{n+1} \) is the spot rate for period \( n+1 \).

The one-year forward rate one year from now is:

\[ f_1 = \left( \frac{(1+s_2)^2}{1+s_1} \right)^{-1} = \left( \frac{(1+0.050252)^2}{1+0.04} \right)^{-1} = 6.060506\% \]

The one-year forward rate two years from now is:

\[ f_2 = \left( \frac{(1+s_3)^3}{1+s_2} \right)^{-1} = \left( \frac{(1+0.060829)^3}{1+0.050252} \right)^{-1} = 8.2303633\% \]

To check your work, the product of one plus the forward rates must equal to the three-year spot rate:

\[ \sqrt[3]{(1+\text{1-year spot rate})(1+\text{1-year forward rate one year from now})(1+\text{1-year forward rate two years from now})} - 1 = (1+\text{3-year spot rate}) \]
Step 3: Diagram the lattice

The simplest lattice is when we assume that the chance of interest rates going up is the same as going down, and the probability of an increase in interest rates is the same as a decrease in interest rates.

The easiest way to view the lattice is to map out the cash flows into the various periods. For a 6 percent, annual-pay bond, the cash flows are $6 each period, plus the maturity value at the end of three years:

```
<table>
<thead>
<tr>
<th>TODAY</th>
<th>PERIOD 1</th>
<th>PERIOD 2</th>
<th>PERIOD 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>CF = $6</td>
<td>CF = $106</td>
<td>CF = $6</td>
<td>CF = $106</td>
</tr>
<tr>
<td>CF = $6</td>
<td>CF = $6</td>
<td>CF = $6</td>
<td>CF = $106</td>
</tr>
<tr>
<td>Value = ?</td>
<td>CF = $6</td>
<td>CF = $6</td>
<td>CF = $106</td>
</tr>
</tbody>
</table>
```

Step 4: Calculate the forward rates considering interest rate volatility

Using a binomial model, where interest rates either go up or go down, we need to make an assumption about the path that interest rates may take and, hence, the volatility of interest rates. If we assume that interest rates follow a lognormal random walk, and we assume that the volatility of interest rates is 5%, this means that the difference between the interest rate when rates go up and the interest rate when rates go down is $2\sigma$, where $\sigma$ is the volatility. If we assume that this is the volatility at every juncture, or node, we can determine the interest rates associated with the various paths.

The interest rate between today (theoretically, the end of period 0) and the end of the first period is the spot rate today, which in our example is 4 percent.

If we make the assumptions that the estimated forward rate is the expected interest rate, we use this forward rate to determine the rate if rates go down and the rate if rates go up by using the volatility and an assumption of continuous compounding:

Rate if the interest rate increases: forward rate x $e^{\sigma} = 0.06060506 \times e^{0.05} = 0.0637123$ or 6.37123%

Rate if the interest rate decreases: forward rate x $e^{-\sigma} = 0.06060506 \times e^{-0.05} = 0.0576493$ or 5.76493%
The ratio of the up and the down interest rates is $e^{2\sigma}$ or 1.10517.

What happens in when we go to the following period? The same logic applies: the forward rate is the expected rate, and we use the volatility measure to estimate the interest rates. What is different is that there are now more interest rates: we need an interest rate for each case: up-up, up-down, down-up, and down-down. With a closed lattice, the up-down and down-up are the same, so we will combine them into one element on the tree.

The interest rates from the end of period 2 to the end of period 3 are:

- Rate after up-up: \(8.2303633\% \cdot e^{0.10} = 9.09596\%
- Rate after up-down or down-up: \(8.2303633\%\)
- Rate after down-down: \(8.2303633\% \cdot e^{-0.10} = 7.44714\%

The ratio of the highest to the lowest rate is $e^{4\sigma}$ or 1.2214. A common way of reporting the interest rates in a lattice is:

\[
\begin{align*}
9.09596\% \\
6.37123\% \\
4.00000\% \\
4.00000\% \\
5.76493\% \\
7.44714\% \\
\end{align*}
\]

Filling out the tree and labeling the different possible paths, the tree is ready for us to calculate values:
Step 5: Calculate the value of the security by discounting through the lattice

We calculate the value of the security by working backward from the end of period 3, discounting using the appropriate one-period forward rate. For example, the value in box 4 is the average of the discounted cash flows from box 7 and box 8. Because the cash flows are the same, the value in box 4 is simply:

\[
\text{Value} = \frac{\$106}{(1+0.09596)} = 97.16217
\]

The value in box 5 is \(\frac{\$106}{(1+0.0823036)} = 97.93924\), and the value in box 6 is \(\frac{\$106}{(1+0.0744714)} = 98.65316\).

We calculate the values in box 2 and 3 by discounting and averaging. The value in box 2 is:

\[
\frac{\$97.16217+6}{(1+0.0637123)} + \frac{\$97.93924+6}{(1+0.0637123)} \approx 97.34841
\]

Continuing this all the way to box 1, we calculate the value of the bond to be $100.

If we arrive at the same value as without the tree, have we accomplished anything? Yes and no. We have not estimated a value different than using other methods because the security does not have an embedded option. Embedded options given either the investor or the issuer a valuable option; these options include a call feature, a put feature, and a conversion feature. What we have done is establish a framework for introducing and valuing a security with an embedded option.

If a bond example is callable, any time the bond’s value according to the lattice exceeds the call price, the value is the assumed to be the call price. If, on the other hand, the bond is putable, each time the value of the bond is less than the put price, the put price is substituted for the bond’s value.
To demonstrate the use of the tree for valuing a security with an embedded option, assume that the 6 percent annual pay bond is putable at $100 in one year. This means that in box 4, the value of the bond is the larger of $100 or the calculated present value (which is $97.16217), and in box 5 the value of the bond is the larger of $100 or the calculated present value (which is $97.93924). This process continues, with a comparison of the present value with the put price.

The tree becomes:

The value of the putable bond is $102.0299.
Index

bootstrapping, 2
embedded option, 2, 7
forward rate, 4

interest rate lattices, 2
interest rate trees, 2
spot rate, 2