PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 100, Number 3, July 1987

ISOMORPHISMS OF GRAPH GROUPS

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ABSTRACT. Given a graph X, define the presentation PX to have generators the vertices of X, and a relation $xy = yx$ for each pair x, y of adjacent vertices. Let GX be the group with presentation PX, and given a field K, let KX denote the K-algebra with presentation PX. Given graphs X and Y and a field K, it is known that the algebras KX and KY are isomorphic if and only if the graphs X and Y are isomorphic. In this paper, we use this fact to prove that if the groups GX and GY are isomorphic, then so are the graphs X and Y.

Given a graph X with vertex set $V(X)$, we define the presentation PX to be that having as generators the elements of $V(X)$, with a defining relation $vw = wv$ for each pair v and w of adjacent vertices of X. PX can be regarded as a presentation of a group GX, or of a K-algebra KX over a field K. In [**1**], Kim, Makar-Limanov, Neggers, and Roush proved that if the algebras KX and KY are isomorphic, then so are the graphs X and Y. (In their formulation, two generators commute provided they are *not* adjacent in the graph. This is sufficient for our purposes, since if two graphs are isomorphic, so are their complements.)

Let K be a field. In this note we will show that if the groups GX and GY are isomorphic, then so are the algebras KX and KY, thus demonstrating:

THEOREM. *If the groups GX and GY are isomorphic, then so are the graphs X and Y.*

Let $f\colon GX \to GY$ be an isomorphism. Denote by G_2X the quotient group $GX/[GX,(GX)']$. Then f induces an isomorphism $f_2\colon G_2X \to G_2Y$. Let $V(X)$ and $V(Y)$ be totally ordered, and denote both orderings by $<$. We will not distinguish by notation between a vertex of X, the corresponding element of GX, and the image of this element in G_2X. For each vertex x of X, $f_2(x)$ can be written uniquely in the form $y_1^{a_1}y_2^{a_2}\cdots y_n^{a_n}C_x$, where C_x is an element of the commutator subgroup $(G_2Y)'$, $y_1 < y_2 < \cdots < y_n$, and the integers a_r are all nonzero. Define $f_*(x) = a_1y_1 + a_2y_2 + \cdots + a_ny_n$. We will show that the function $f_*\colon X \to KY$ extends to a homomorphism $f_*\colon KX \to KY$ by showing that if $xx' = x'x$ is a relation of PX, then $f_*(x)f_*(x') = f_*(x)f_*(x')$ in KY.

LEMMA. *The commutators $\{[x_i, x_j] \mid x_i < x_j$ and x_i and x_j are not adjacent in $X\}$ of G_2X are linearly independent.*

PROOF. Consider the exact sequence

$$1 \to N \to FX \to GX \to 1$$

Received by the editors January 21, 1985 and, in revised form, May 12, 1986.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 20F05; Secondary 20F12.

associated with the presentation PX. Since N is a subgroup of FX', there is an exact sequence

$$1 \to \hat{N} \to (F_2X)' \to (G_2X)' \to 1$$

where $\hat{N}$ denotes the image of N in F_2X. By the Basis Theorem, $(F_2X)'$ is a free abelian group with a basis consisting of the commutators $\{[x_i, x_j] \mid x_i < x_j\}$. Since $\hat{N}$ is the subgroup of $(F_2X)'$ generated by the collection $\{[x_i, x_j] \mid x_i < x_j, x_i$ and x_j adjacent$\}$, the assertion follows.

Suppose that x and x' are adjacent vertices of X. Let $f_2(x) = y_1^{a_1} y_2^{a_2} \cdots y_n^{a_n} C_x$ and $f_2(x') = y_1^{b_1} y_2^{b_2} \cdots y_n^{b_n} C_{x'}$, with $y_1 < y_2 < \cdots < y_n$, where we allow some of the exponents a_r and b_s to be 0 so that we may use the same elements of $V(Y)$ in both expressions. Writing $(G_2Y)'$ additively, and recalling that in G_2Y, commutators are central, we have

$$\begin{aligned} 0 = [f_2(x), f_2(x')] &= \sum_{r,s} [y_r^{a_r}, y_s^{b_s}] \\ &= \sum_{r,s} a_r b_s [y_r, y_s] = \sum_{r<s} (a_r b_s - a_s b_r)[y_r, y_s]. \end{aligned}$$

Thus, by the Lemma, if y_r and y_s are nonadjacent vertices of Y, then $a_r b_s - a_s b_r = 0$. Clearly, $f_*(x) = \sum_r a_r y_r$ and $f_*(x') = \sum_s b_s y_s$, so that

$$f_*(x) f_*(x') - f_*(x') f_*(x) = \sum_{r,s} (a_r b_s - b_r a_s) y_r y_s.$$

If y_r and y_s are adjacent vertices of Y, then $y_r y_s = y_s y_r$ in KY, so the net coefficient of $y_r y_s$ in this sum is $(a_r b_s - b_r a_s) + (a_s b_r - b_s a_r) = 0$. If y_r and y_s are nonadjacent vertices of Y, then the net coefficient of $y_r y_s$ is $a_r b_s - b_r a_s$, and we saw above that this must by 0 when y_r and y_s are not adjacent. Finally, for each r, the coefficient of y_r^2 is $a_r b_r - b_r a_r = 0$. Thus, $f_*(x) f_*(x') - f_*(x') f_*(x) = 0$, so f_* extends to a homomorphism from KX to KY.

It is now an easy matter to check that for each vertex x of X, $(f^{-1})_* f_*(x) = x$, and for each vertex y of Y, $f_*(f^{-1})_*(y) = y$, so that f_* is in fact an isomorphism.

The following shorter proof of our theorem has been pointed out by an anonymous reviewer: Define $G_0 = GX$ and $G_{n+1} = [G_n, G]$ for $n \geq 0$, and let LX denote the Lie algebra $K \otimes \sum G_n / G_{n+1}$. Then the algebra KX is the universal enveloping algebra of LX, so that if GX and GY are isomorphic groups, then the Lie algebras LX and LY, and hence the algebras KX and KY, are isomorphic.

REFERENCES

1. K. H. Kim, L. Makar-Limanov, J. Neggers and F. Roush, *Graph algebras*, J. Algebra **64** (1980), 46–51.

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