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Vol. 110, No. 2, October 15, 1987
*Printed in Belgium*

# Subgroups of Graph Groups

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*Communicated by Barbara L. Osofsky*

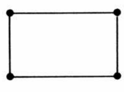
Received January 31, 1986

Given a graph $X$, we define a group $GX$ as follows: $GX$ is generated by the vertices of $X$, with a defining relation $xy = yx$ for each pair $x$, $y$ of vertices joined by an edge of $X$. A group is called a graph group if it is isomorphic to $GX$ for some graph $X$. We will not distinguish by notation in what follows between a vertex of $X$ and the corresponding element of $GX$.

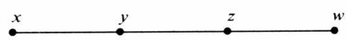
We remark that the "extreme" cases of graphs, namely the complete and the completely disconnected graphs, correspond, respectively, to free abelian and free groups. It is well known that any subgroup of a group of either of these types is again of the same type, and it is thus natural to ask to what extent and in what form this is true of the graph groups that lie "in-between" these extreme cases.

In this article we will prove the following:

THEOREM. *Let $X$ be a finite graph. Then every subgroup of the graph group $GX$ is itself a graph group if and only if no full subgraph of $X$ has either of the two forms*



$C_4$




$L_3$

Certainly the theorem holds for the (unique) graph with one vertex. We next demonstrate the sufficiency of the stated condition for graphs with more than one vertex.

Suppose first that $X$ is not connected, say the components of $X$ are $X_1, ..., X_n$. Then $GX$ is the free product of the groups $GX_1, ..., GX_n$. If $X$ has no subgraphs isomorphic to either $C_4$ or $L_3$, then neither has any of the graphs $X_i$, so by induction, any subgroup of any of the groups $GX_i$ is a



0021-8693/87 $3.00

graph group. But any subgroup of $GX$ is isomorphic to the free product of a free group and certain conjugates in $GX$ of subgroups of the $GX_i$. Since both free groups and free products of graph groups are graph groups, so is every subgroup of $GX$.

Now suppose $X$ is connected.

LEMMA. *If $X$ is a finite connected graph with no full subgraph isomorphic to either $C_4$ or $L_3$, then there is at least one vertex of $X$ which is joined to every other vertex of $X$.*

*Proof.* If $X$ is a complete graph, this is clear. Suppose $X$ is not complete. Let $V$ be the vertex set of $X$, and let $S$ be any minimal separating set of vertices of $X$. Since $X$ is connected, $S$ is nonempty. Clearly, every circuit of $X$ of length at least 4 has a chord, so by [2, Solution to Problem 9.29b], $S$ induces a complete subgraph of $X$. Let $x$ be any vertex in $S$. We will show that if $y$ lies in $V-S$, then $x$ and $y$ are adjacent. Let $X_{(S)}$ denote the graph obtained from $X$ by deleting the vertices in $S$ and all edges which have an endpoint in $S$. Then $X_{(S)}$ is not connected. Let $C_y$ denote the component of $X_{(S)}$ containing $y$, and let $C$ be any other component of $X_{(S)}$. Since $S$ is a *minimal* separating set, there must be a vertex $y'$ in $C_y$ adjacent to $x$, and a vertex $z$ in $C$ adjacent to $x$. Suppose $x$ and $y$ are not adjacent. Since $C_y$ is connected, we may suppose that $y$ and $y'$ are adjacent. But then the subgraph of $X$ induced by the vertices $x$, $y$, $y'$, and $z$ is isomorphic to $L_3$, a contradiction.

Let $z$ be any vertex of $X$ which is adjacent to all the other vertices of $X$, and let $X_{(z)}$ denote the graph obtained from $X$ by deleting the vertex $z$ and all edges of $X$ which have $z$ as an endpoint. Then $z$ is in the center of $GX$, so $GX=\langle z\rangle \times GX_{(z)}$. Let $p: GX \to GX_{(z)}$ denote projection onto the second summand. Then there is an exact sequence

$$1 \longrightarrow \langle z\rangle \longrightarrow GX \xrightarrow{p} GX_{(z)} \longrightarrow 1.$$

Let $H$ be a subgroup of $GX$, and consider the corresponding exact sequence

$$1 \longrightarrow H \cap \langle z\rangle \longrightarrow H \xrightarrow{p} pH \longrightarrow 1.$$

We will show that this sequence splits. By induction, $pH$ is a graph group (since $X_{(z)}$ has fewer vertices than $X$, and $X_{(z)}$ has no subgraph of either of the "forbidden" forms), say $pH=GY$ for some graph $Y$. Given any $h \in H$, $p(h)\,h^{-1} \in \langle z\rangle$, so for each vertex $y$ of $Y$, there is an element of the form $yz^n$ in $H$. Define $i(y)=yz^n$. To see that this indeed defines a homomorphism, suppose that $y$ and $y'$ are adjacent vertices of $Y$. Then $[i(y), i(y')]=[yz^n, y'z^{n'}]=[y, y']=1$, since $z$ is in the center of $GX$.

Clearly, $pi = 1_{pH}$. But $H \cap \langle z \rangle$ is in the center of $H$, so $H = H \cap \langle z \rangle \times pH$. We claim that $H$ is a graph group. If $H \cap \langle z \rangle$ is trivial, this is clear. If $H \cap \langle z \rangle$ is infinite cyclic, then $H = GY'$, where $Y'$ is the graph obtained from $Y$ by appending a new vertex which is adjacent to all the other vertices of $Y$.

We now demonstrate the necessity of the given condition. It will suffice to find non-graphic subgroups of the groups $GC_4$ and $GL_3$, since if $Y$ is a full subgraph of $X$, then the subgroup of $GX$ generated by the vertices of $Y$ is isomorphic to $GY$.

$GC_4$ is isomorphic to the direct product of two free groups of rank two. This group is not coherent [4]; that is, $GC_4$ has a subgroup which is finitely generated but not finitely related. Clearly this subgroup cannot be a graph group.

Let $t$ denote the generator of the group $Z_2$, and let $K$ denote the kernel of the homomorphism $s: GL_3 \to Z_2$ defined by setting $s(v) = t$ for each vertex $v$ of $L_3$. We will show that $K$ is not a graph group. To do this, we will need the following observations: let $X$ be a finite graph and let $G = GX$. There is an exact sequence, due to Dicks [1]

$$0 \to (ZG)^{d_n} \to (ZG)^{d_{n-1}} \to \cdots \to (ZG)^{d_0} \to Z \to 0$$

of left $ZG$-modules, where $d_i$ is the number of $i$-vertex complete subgraphs of $X$ ($d_0 = 1$). Thus, $G$ has Euler characteristic $\chi(GX) = \sum (-1)^i d_i$. Note that $d_1$ is the number of vertices of $X$ and $d_2$ is the number of edges. If $X$ is a tree, then $d_i = 0$ for all $i > 2$, so $\chi(GX) = 1 - d_1 + d_2 = 0$, by a well-known fact about trees.

The other fact which follows from Dicks' sequence is that the cohomological dimension of $GX$, $\mathrm{cd}(GX)$, is equal to the size of the largest complete subgraph of $X$. In particular, if $X$ is a tree, then $\mathrm{cd}(GX) = 2$.

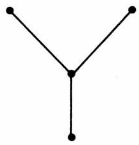
Let us apply these facts to the subgroup $K$ of $G = GL_3$. $K$ has index two in $G$, so the above sequence is also a resolution of $Z$ by finitely generated free $K$-modules. Thus, $\mathrm{cd}(K) \leqslant \mathrm{cd}(G) = 2$. Also, by Schanuel's Lemma, the Euler characteristic is independent of the choice of finitely generated resolution, so because $ZG$ is a free $ZK$-module of rank two, $\chi(K) = 2\chi(G) = 0$.

Now, a straightforward (but tedious) application of the Reidemeister-Schreier technique shows that $K$ has the presentation

$$\langle a, b, c, d \mid ab = ba, bc = cd, bc^2 d = dbc^2 \rangle$$

(The generators $a$, $b$, $c$, and $d$ correspond, respectively, to the elements $x^{-1}y$, $y^2$, $y^{-1}z$, and $z^{-1}w$ of $K$.)

Suppose that $K$ is a graph group, say $K = GY$. Then, since $K/K'$ is free abelian of rank 4, $Y$ must have 4 vertices. Also, since $\mathrm{cd}(K) \leqslant 2$, $Y$ has no triangles. This, together with the fact that $\chi(K)=0$ implies that $Y$ is a tree. There are only two trees with 4 vertices,



and

a b

To rule out graph (a), we note that $K$ has trivial center. This follows from the fact that $K$ is isomorphic to the free product of the groups $\langle a, b, c \mid ab=ba, bc=cb \rangle$ and $\langle c', d \mid c'd=dc' \rangle$ amalgamating the infinite cyclic subgroups generated by $bc^2$ (resp. $c'$). Since no power of $bc^2$ commutes with $a$, [3, Corollary 4.5] implies that $K$ has trivial center.

We will rule out graph (b) by proving that $K$ is not isomorphic to $GL_3$. Denote by $K_2$ the quotient $K/[K, K']$ and by $G_2$ the quotient $GL_3/[GL_3, GL_3']$. In $K_2$, $1=[bc^2, d]=[b, d][c, d]^2$, so that, modulo $(K_2')^2$, the image of $b$ is central. Suppose the element $g$ in $G_2$ is central modulo $(G_2')^2$, say $g=x^p y^r z^s w^t C$, where $C \in G_2'$.

Then $[x, g]=[x, z]^s [x, w]^t \in (G_2')^2$. But $G_2'$ is a free abelian group generated by the basic commutators $[x, w]$, $[x, z]$, and $[y, w]$, so $s$ and $t$ must be even. Considering $[w, g]$ shows that $p$ and $r$ must also be even. Thus, $g$ is a square modulo $G_2'$. But $K_2/K_2'=K/K'$ is a free abelian group generated by the images of $a, b, c$, and $d$, so that $b$ cannot be a square modulo $K_2'$. Thus, $K$ and $GL_3$ are not isomorphic, so $K$ is not a graph group.

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