Reprinted from JOURNAL OF ALGEBRA
All Rights Reserved by Academic Press, New York and London

Vol. 106, No. 2, April 1, 1987
Printed in Belgium

Graph Groups, Coherence, and Three-Manifolds

CARL DROMS

Department of Mathematics and Computer Science, James Madison University, Harrisonburg, Virginia 22807

Communicated by Barbara L. Osofsky

Received May 8, 1985

I. INTRODUCTION

We study groups given by presentations each of whose defining relations is of the form $xy = yx$ for some generators x and y. To such a presentation we associate a graph X whose vertices are the generators, two vertices x and y being adjacent in X if and only if $xy = yx$ is a defining relation.

Given a graph X, we denote by GX the group defined by the presentation associated to X in this way. We call GX a graph group. These groups have been studied by Kim and Roush [8], and by Dicks [3].

In this paper we prove the following:

THEOREM 1. *If X is a finite graph, then the group GX is coherent if and only if each circuit of X of length greater than three has a chord.*

(Recall that a group is called coherent if each of its finitely generated subgroups is finitely presented.)

THEOREM 2. *If X is a finite graph, then the group GX is the fundamental group of a three-dimensional manifold if and only if each connected component of X is either a tree or a triangle.*

II. GRAPH-THEORETIC TERMINOLOGY

We refer the reader to [9] for terminology in graph theory not defined here. A full subgraph U of a graph X is a graph whose vertex set is a subset of the vertex set of X, two vertices being adjacent in U if and only if they are adjacent in X. Since each full subgraph of X is determined by its vertex set, we call U the subgraph of X induced by its vertex set. We denote by $\langle S \rangle$ the subgraph of X induced by the subset S of the vertices of X. Note

0021-8693/87 $3.00

that the subgroup of GX generated by the elements of S is isomorphic to $G\langle S\rangle$.

A vertex x of the graph X will be called central if x is adjacent to all the other vertices of X.

III. Group-Theoretic Preliminaries

Let X be a finite graph. Given an element $g \in GX$, with $g = x_1^{e_1} x_2^{e_2} \cdots x_k^{e_k}$, where each x_i is a vertex of X, we define

$$|g| = e_1 + e_2 + \cdots + e_k.$$

$|g|$ is independent of the expression of g as a product of powers of generators, since each relator has exponent sum 0. Let $KX = \{g \in GX\colon |g| = 0\}$. Clearly KX is a subgroup of GX.

If U and V are full subgraphs of X, with $X = U \cup V$ and $W = U \cap V$, then $GX = GU *_{GW} GV$, as follows easily by examining generators and relations. In particular, if $U \cap V$ is empty, then $GX = GU * GV$. Since free products of 3-manifold groups are 3-manifold groups [4, Lemma 3.2], and free products of coherent groups are coherent [7, Theorem 8], it will suffice to prove Theorems 1 and 2 for connected graphs.

Proposition. *Let X be a finite connected graph, and let U and V be full subgraphs of X with $X = U \cup V$ and $W = U \cap V$. Then*

$$KX = KU \underset{KW}{*} KV$$

Proof. Since $GX = GU *_{GW} GV$, [11, Theorem 13] implies that GX acts on a directed tree Y, whose vertices are the left cosets of the subgroups GU and GV in GX, and whose edges are the left cosets of GW in GX. Thus KX acts on Y also. In fact, KX acts transitively on the edges of Y; to see this, let w be any vertex of W, and let gGW be any edge of Y. Then $w^{|g|}g^{-1} \in KX$, and $(w^{|g|} g^{-1})(gGW) = w^{|g|}GW = GW$, so there is only one orbit of edges under the action of KX. Since the vertices GU and GW lie in different orbits of the KX-action on Y, the quotient directed graph Y/KX consists of two vertices joined by an edge, so again by [11, Theorem 13],

$$KX = KX \cap GU \underset{KX \cap GW}{*} KX \cap GV = KU \underset{KW}{*} KV.$$

Corollary. *Let T be a finite tree with $n+1>0$ vertices. Then KT is a free group of rank n. Further, KT is freely generated by a set of elements*

$k_1, k_2, \ldots, k_n$ in one-to-one correspondence with the n edges of T; the generator corresponding to the edge joining the vertices x and y may be chosen equal to either $x^{-1}y$ or $y^{-1}x$.

Proof. This is clear if $n=0$ or if $n=1$. If $n>1$, choose a pendent vertex x of T, let y be the unique vertex of T adjacent to x, and let T' denote the tree obtained from T by deleting the vertex x and the edge joining x and y. Then $GT = GT' *_{G\langle y\rangle} G\langle x, y\rangle$, so by the above proposition, $KT = KT' *_{K\langle y\rangle} K\langle x, y\rangle$. By induction, KT' is free of rank $n-1$. Clearly $K\langle x, y\rangle$ is infinite cyclic, and $K\langle y\rangle$ is trivial, so KT is free of rank n. The assertion about generators follows from induction and the fact that each of $x^{-1}y$ and $y^{-1}x$ generates $K\langle x, y\rangle$.

IV. Proof of Theorem 1

Suppose every circuit of X of length greater than three has a chord. If X is complete, then GX is finitely generated free abelian, and so coherent. Otherwise, X has a separating set A of vertices which induces a complete subgraph of X [9, Solution to Problem 9.29b]. That is, there are proper full subgraphs X_1 and X_2 of X such that $X = X_1 \cup X_2$, $\langle A\rangle = X_1 \cap X_2$, and $\langle A\rangle$ is complete. Thus,

$$GX = GX_1 \underset{G\langle A\rangle}{*} GX_2$$

Every circuit of either X_1 or X_2 of length greater than three has a chord, so by induction, GX_1 and GX_2 are coherent. $G\langle A\rangle$ is finitely generated free abelian, so by [7, Theorem 8], GX is also coherent.

Now suppose that the graph X is a circuit of length greater than three and let x and y be two nonadjacent vertices of X. Then there are proper full subgraphs X_1 and X_2 of X such that $X = X_1 \cup X_2$, $X_1 \cap X_2 = \langle x, y\rangle$, and X_1 and X_2 are trees. Thus,

$$KX = KX_1 \underset{K\langle x,y\rangle}{*} KX_2.$$

Each of KX_1 and KX_2 is a finitely generated free group, so KX is finitely generated. $K\langle x, y\rangle$ is the normal closure in the free group $G\langle x, y\rangle$ of $x^{-1}y$, so $K\langle x, y\rangle$ is not finitely generated. By [1], KX is not finitely presented, so GX is not coherent. It follows that if some circuit of X of length greater than 3 has no chord, then GX has a noncoherent subgroup, and is thus itself not coherent.

V. An Ordering of the Vertices of a Tree

Let T be a finite tree, and let x_0 be a pendent vertex of T. We will describe a linear ordering of the vertices of T which we will use in the proof of Theorem 2. Given a vertex y, denote by $\text{star}^+(y)$ the set $\{z: y$ lies on the path joining x_0 to $z\}$. This is well defined since T is a tree. Define $\text{star}(y)$ to be the set of vertices in $\text{star}^+(y)$ which are adjacent to y. We order the vertices of T as follows: first, for each vertex y, arbitrarily order the set $\text{star}(y)$. Then, given 2 vertices y and z of T, set $y<z$ if either of the two conditions below is satisfied:

(i) $z \in \text{star}^+(y)$,

(ii) there are vertices v, y_0, and z_0 with $y_0, z_0 \in \text{star}(v)$, $y \in \text{star}^+(y_0)$, $z \in \text{star}^+(z_0)$, and $y_0 < z_0$ in the ordering chosen on $\text{star}(v)$.

$<$ is a linear ordering of the vertices of T, since T is a tree.

VI. Proof of Theorem 2

By [10], any 3-manifold group is coherent, so we need only consider connected graphs in which every circuit of length greater than three has a chord.

If the graph X is a triangle, then $GX = \pi_1(S^1 \times S^1 \times S^1)$, so GX is a 3-manifold group. Let T be a finite tree and let x_0 be a pendent vertex of T. We will show that GT is a three-manifold group. Let $s: GT \to gp\langle x_0 \rangle$ be the homomorphism determined by setting $s(y) = x_0$ for each vertex y of T. Then $\ker(s) = KT$, so there is a split exact sequence

$$1 \longrightarrow KT \longrightarrow GT \xrightarrow{\ s\ } gp\langle x_0 \rangle \longrightarrow 1.$$

Thus, GT is isomorphic to the semidirect product $KT \ltimes gp\langle x_0 \rangle$.

To show that GT is a 3-manifold group, we will use a different generating set for KT than that described above. Let $<$ denote an ordering of the vertices of T as defined in section III. Given a vertex x other than x_0, set $\hat{x} = y^{-1}x$, where y is the unique vertex of T for which $x \in \text{star}(y)$. By the above corollary the set $\{\hat{x} \mid x \neq x_0\}$ freely generates KT. For $x \neq x_0$, let $x^* = \hat{x}\,\hat{x}_k^{-1}\,\hat{x}_{k-1}^{-1} \cdots \hat{x}_1^{-1}$, where $\text{star}(x) = \{x_1, x_2, \ldots, x_k\}$ and $x_1 < x_2 < \cdots < x_k$. (If $\text{star}(x)$ is empty, we define $x^* = \hat{x}$.) A routine computation shows that if $\text{star}^+(x) = \{x_1, x_2, \ldots, x_m\}$, with $x_1 < x_2 < \cdots < x_m$, then $x^* x_1^* x_2^* \cdots x_m^* = \hat{x}$. Thus, the set $\{x^* \mid x \neq x_0\}$ also freely generates KT. Let $a: KT \to KT$ be the automorphism defined by $a(k) = x_0^{-1} k x_0$ for each $k \in KT$. If x is a vertex of T other than x_0, then the elements x^* and x

of GT commute, so that $a(x^*) = (x^{-1}x_0)^{-1}x^*(x^{-1}x_0)$. Therefore, since $x^{-1}x_0 \in KT$, $a(x^*)$ is conjugate in KT to x^*. Furthermore, if the vertex set of T is the set $\{x_0, x_1, ..., x_n\}$, with $x_0 < x_1 < \cdots < x_n$, then $x_1^* x_2^* \cdots x_n^* = \hat{x}_1 = x_0^{-1}x_1$. Because T is connected, the vertices x_0 and x_1 must be adjacent, so

$$a(x_1^* x_2^* \cdots x_n^*) = x_0^{-1}(x_0^{-1}x_1)\, x_0 = x_0^{-1}x_1 = x_1^* x_2^* \cdots x_n^* .$$

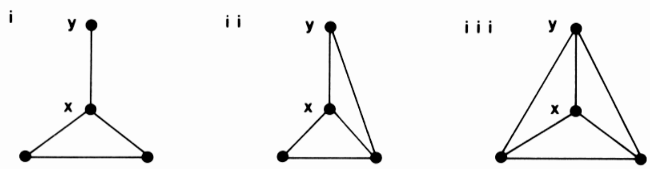
Let D_n^2 denote the space obtained by removing n interior points from the disk D^2. Then $\pi_1(D_n^2) = KT$ and by [2, Theorem 1.10], there is a homeomorphism h of D_n^2 which fixes the boundary of D^2 pointwise and for which $a = h_*$, the automorphism of $\pi_1(D_n^2)$ induced by h. Let $\sim$ be the least equivalence relation on the space $D_n^2 \times [0, 1]$ for which $[p, 0] \sim [h(p), 1]$, and let $M = D_n^2 \times [0, 1]/\sim$. Clearly, M is a 3-manifold. The fundamental group of M is isomorphic to the semidirect product $\pi_1(D_n^2) \ltimes Z$, where Z is an infinite cyclic group with generator t, and, for each $g \in \pi_1(D_n^2)$, $t^{-1}gt = h_*(g)$ [2, proof of Theorem 2.2]. Since $h_* = a$, this group is isomorphic to GT.

To complete the proof of Theorem 2, we shall need the following:

LEMMA. *Let X be a finite graph with central vertex x, and suppose that GX is a 3-manifold group. If y is any vertex of X other than x, then the graph Y obtained from X by deleting the vertices x and y is totally disconnected.*

Proof. Since X is finite, GX is finitely generated, so by [6], GX is the fundamental group of a compact 3-manifold. Let X' be the graph obtained from X by deleting the vertex x. Then GX' is a normal subgroup of GX with infinite cyclic quotient, so by [12], GX' is the fundamental group of a surface. GY is a subgroup of infinite index in GX', so by [5], GY is free. Thus, Y must be totally disconnected, since otherwise, GY would have a free abelian subgroup of rank two.

Now, suppose that the graph X is neither a tree nor a triangle, and that every circuit of X of length greater than three has a chord. Then X must have an induced subgraph of one of the following forms:



It follows from the lemma that none of the graph groups associated with these graphs is a 3-manifold group. Because every subgroup of the fundamental group of a 3-manifold is itself a 3-manifold group, GX is not a 3-manifold group.

References

1. G. Baumslag, A remark on generalized free products, *Proc. Amer. Math. Soc.* **13** (1962), 53–54.
2. J. Birman, "Braids, Links, and Mapping Class Groups," Ann. Math. Stud., No. 82, Princeton Univ. Press, Princeton, N.J., 1974.
3. W. Dicks, An exact sequence for rings of polynomials in partly commuting indeterminates, *J. Pure Appl. Algebra* **22** (1981), 215–228.
4. J. Hempel, "3-Manifolds," Ann. Math. Stud., No. 86, Princeton Univ. Press, Princeton, N.J., 1976.
5. A. Hoare, A. Karrass, D. Solitar, Subgroups of infinite index in fuchsian groups, *Math. Z.* **125** (1972), 59–69.
6. W. Jaco, Finitely presented subgroups of three-manifold groups, *Invent. Math.* **13** (1971), 335–346.
7. A. Karrass, D. Solitar, The subgroups of a free product of two groups with an amalgamated subgroup, *Trans. Amer. Math. Soc.* **150** (1970), 227–255.
8. K. H. Kim, F. Roush, Homology of certain algebras defined by graphs, *J. Algebra* **64** (1980), 179–186.
9. L. Lovasz, "Combinatorial Problems and Exercises," North-Holland, Amsterdam, 1979.
10. G. Scott, Finitely generated 3-manifold groups are finitely related, *J. London Math. Soc.* (*2*) **6** (1973), 437–440.
11. J.-P. Serre, "Arbres, amalgames et SL_2," Asterisque 46, Soc. Math. de France, Paris, 1977.
12. J. Stallings, On fibering certain three-manifolds, *in* "Topology of Three Manifolds," pp. 95–100, Preentice–Hall, Englewood Cliffs, N.J., 1962.

Printed by the St. Catherine Press Ltd., Tempelhof 41, Bruges, Belgium